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On deformations of 2d SCFTs

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Abstract: Motivated by the representation of the super Virasoro constraints as generalized Dirac-Kähler constraints $(\mathbf{d} \pm \mathbf{d}^\dagger)\,|\psi\rangle = 0$ on loop space, examples of the most general continuous deformations $\mathbf{d} \to e^{-\mathbf{W}}\,\mathbf{d}\,e^{\mathbf{W}}$ are considered which preserve the superconformal algebra at the level of Poisson brackets. The deformations which induce the massless NS and NS-NS backgrounds are exhibited. Hints for a manifest realization of S-duality in terms of an algebra isomorphism are discussed.
It is shown how the first order theory of 'canonical deformations' is reproduced and how the deformation operator $\mathbf{W}$ encodes vertex operators and gauge transformations.

Keywords: Superstrings and Heterotic Strings, String Duality, Conformal Field Models in String Theory, Differential and Algebraic Geometry.

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1. Introduction

Supersymmetric field theories look like Dirac-Kähler systems when formulated in Schrödinger representation. This has been well studied in the special limits where only a finite number of degrees of freedom are retained, such as the semi-classical quantization of solitons in field theory (see e.g. [1] for a brief introduction and further references). That this phenomenon is rooted in the general structure of supersymmetric field theory has been noted long ago in the second part of [2] (see also the second part of [3]). For 2 dimensional superconformal field theories describing superstring worldsheets a way to exploit this fact for the construction of covariant target space hamiltonians (applicable to the computation of curvature corrections of string spectra in nontrivial backgrounds) has been proposed in [4]. In the construction of these hamiltonians a pivotal role is played by a new method for obtaining functional representations of superconformal algebras (corresponding to nontrivial target space backgrounds) by means of certain deformations of the superconformal algebra.

In [4] the focus was on deformations which induce Kalb-Ramond backgrounds and only the 0-mode of the superconformal algebra was considered explicitly (which is sufficient for the construction of covariant target space hamiltonians). Here this deformation technique is developed in more detail for the full superconformal algebra and for all massless bosonic string background fields. Other kinds of backgrounds can also be incorporated in principle and one goal of this paper is to demonstrate the versatility of the new deformation technique for finding explicit functional realizations of the two-dimensional superconformal algebra.

The setting for our formalism is the representation of the superconformal algebra on the exterior bundle over loop space (the space of maps from the circle into target space) by means of K-deformed exterior (co)derivatives $\mathbf{d}_K$, $\mathbf{d}^\dagger{}_K$, where K is the Killing vector field on loop space which induces loop reparameterizations.

The key idea is that the form of the superconformal algebra is preserved under the deformation[1]

$$\begin{aligned} \mathbf{d}_K &\to e^{-\mathbf{W}}\, \mathbf{d}_K\, e^{\mathbf{W}} \\ \mathbf{d}^\dagger{}_K &\to e^{\mathbf{W}^\dagger}\, \mathbf{d}^\dagger{}_K\, e^{-\mathbf{W}^\dagger} \end{aligned} \tag{1.2}$$

if $\mathbf{W}$ is an even graded operator that satisfies a certain consistency condition.

The canonical (functional) form of the superconformal generators for all massless NS and NS-NS backgrounds can neatly be expressed this way by deformation operators $\mathbf{W}$

[1]Throughout this paper we use the term "deformation" to mean the operation (1.2) on the superconformal generators, the precise definition of which is given in section 3.2 (p. 14). These "deformations" are actually *isomorphisms* of the superconformal algebra, but affect its representations in terms of operators on the exterior bundle over loop space. In the literature one finds also other usages of the word "deformation" in the context of superalgebras, for instance for describing the map where the superbrackets $[\cdot,\cdot]_\iota$ are transformed as

$$[A,B]_\iota \to [A,B]_\iota + \sum_{t=1}^{\infty} \omega_i(A,B)\, t^i \tag{1.1}$$

with $\omega_i(A,B)$ elements of the superalgebra and t a real number (see [5]).

that are bilinear in the fermions, as will be shown here. It turns out that there is one further bilinear in the fermions which induces a background that probably has to be interpreted as the RR 2-form.

It is straightforward to find further deformation operators and hence further backgrounds. While the normal ordering effects which affect the superconformal algebras and which would give rise to equations of motion for the background fields are not investigated here, there is still a consistency condition to be satisfied which constrains the admissible deformation operators.

This approach for obtaining new superconformal algebras from existing ones by applying deformations is similar in spirit, but rather complementary, to the method of '*canonical deformations*' studied by Giannakis, Evans, Ovrut, Rama and others [6]–[9]. There, the superconformal generators T and G of one chirality are deformed to lowest order as

$$\begin{aligned} T(z) &\to T(z) + \delta T(z) \\ G(z) &\to G(z) + \delta G(z) \;. \end{aligned} \tag{1.3}$$

Requiring the deformed generators to satisfy the desired algebra to first order shows that δT and δG must be bosonic and fermionic components of a weight 1 worldsheet superfield. (An adaption of this procedure to deformations of the BRST charge itself is discussed in [10]. Another related discussion of deformations of BRST operators is given in [11].)

The advantage of this method over the one discussed in the following is that it operates at the level of quantum SCFTs and has powerful CFT tools at its disposal, such as normal ordering and operator product expansion. The disadvantage is that it only applies perturbatively to first order in the background fields, and that these background fields always appear with a certain gauge fixed.

On the other hand, the deformations discussed here which are induced by $\mathbf{d}_K \to e^{-\mathbf{W}}\mathbf{d}_K e^{\mathbf{W}} \sim e^{-\mathbf{W}}(iG + \bar{G})e^{\mathbf{W}}$ preserve the superconformal algebra for arbitrarily large perturbations $\mathbf{W}$. The drawback is that normal ordering is non-trivially affected, too, and without further work the resulting superconformal algebra is only available on the level of (bosonic and fermionic) Poisson brackets.

We show in section 3.4.2 (p. 24) that when restricted to first order the deformations that we are considering reproduce the theory of canonical deformations (1.3).

Our deformation method is also technically different from but related to the *marginal deformations* of conformal field theories (see [12] for a review and further references), where one sends the correlation function $\langle A\rangle$ of some operator A to the deformed correlation function

$$\langle A\rangle^\lambda := \left\langle A \exp\left(\sum_i \lambda_i \int \mathcal{O}_i \;\mathrm{dvol}\right)\right\rangle , \tag{1.4}$$

where $\mathcal{O}_i$ are fields of conformal weight 1. This corresponds to adding the integral over a field of unit weight to the action. How this relates to the algebraic deformations of the superconformal algebra considered here is discussed in section 3.4.1 (p. 21).

The method discussed here generalizes the transformations studied in [13], where strings are regarded from the non-commutative geometry perspective. The main result of this approach (which goes back to [14] and [15]) is that T-duality as well as mirror symmetry can nicely be encoded by means of automorphisms of the vertex operator algebra. In terms of the above notation such automorphisms correspond to deformations induced by *anti-hermitean* $\mathbf{W}^\dagger = -\mathbf{W}$, which induce pure gauge transformations on the algebra.

The analysis given here generalizes the approach of [13] in two ways: first, the use of hermitean $\mathbf{W}$ in our formalism produces backgrounds which are not related by string dualities. Second, by calculating the functional form of the superconformal generators for these backgrounds we can study the action of anti-hermitean $\mathbf{W}$ on these more general generators and find the transformation of the background fields under the associated target space duality.

In particular, we find a duality transformation which changes the sign of the dilaton and interchanges B- and C-form fields. It would seem that this must hence be related to S-duality. This question requires further analysis.

The structure of this paper is as follows: in section 2 some technical preliminaries necessary for the following discussion are given. The functional loop space notation is introduced in section2.1, some basic facts about loop space geometry are discussed (section 2.2), the exterior derivative and coderivative on that space are introduced (section 2.2.2), and some remarks on isometries of loop space are given in section 2.2.3.

This is then applied in section 3 to the general analysis of deformations of the superconformal generators. First of all, the purely gravitational target space background is shown to be associated to the ordinary K-deformed loop space exterior derivative (section 3.1). Section 3.2 then discusses how general continuous classical deformations of the superconformal algebra are obtained. As a first application, section 3.3.1 shows how this can be used to get the previously discussed superconformal generators for purely gravitational backgrounds from those of flat space by a deformation.

Guided by the form of this deformation the following sections systematically list and analyze the deformations which are associated with the Kalb-Ramond, dilaton, and gauge field backgrounds (sections 3.3.2–3.3.4). It turns out (section 3.3.5) that one further 2-form background can be obtained in a very similar fashion, which apparently has to be interpreted as the S-dual coupling of the D-string to the C_2 2-form background.

After having understood how the NS-NS backgrounds arise in our formalism we turn in section3.4 (p. 21) to a comparison of the method presented here with the well-known 'canonical deformations', which are briefly reviewed in section3.4.1 (p. 21). In section 3.4.2 (p. 24) it is shown how these canonical deformations are reproduced by means of the methods discussed here and how our deformation operator $\mathbf{W}$ relates to the vertex operators of the respective background fields.

Next the inner relations between the various deformations found are further analyzed in section 4. First of all section 4.1 demonstrates how $\mathbf{d}_K$-exact deformation operators yield target space gauge transformations. Then, in section 4.2 the well known realization of T-duality as an algebra isomorphism is adapted to the present context, and in section 4.2.2 the action of a target space duality obtained from a certain modified algebra isomorphism on the various background fields is studied. It turns out that there are certain similarities to the action of loop space Hodge duality, which is discussed in section 4.3.

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Finally a summary and disucssion is given in section 5. The appendix lists some results from the canonical analysis of the D-string action, which are needed in the main text.

2. Loop space

In this section the technical setup is briefly established. The 0-mode $\mathbf{d}_K$ of the sum of the left- and the rightmoving supercurrents is represented as the K-deformed exterior derivative on loop space. Weak nilpotency of this K-deformed operator (namely nilpotency up to reparameterizations) is the essential property which implies that the modes of $\mathbf{d}_K$ and its adjoint generate a superconformal algebra. In this sense the loop space perspective on superstrings highlights a special aspect of the super Virasoro constraint algebra which turns out to be pivotal for the construction of classical deformations of that algebra.

The kinematical configuration space of the closed *bosonic* string is loop space $\mathcal{LM}$, the space of parameterized loops in target space $\mathcal{M}$. As discussed in [4, section 2.1] the kinematical configuration space of the closed *superstring* is therefore the superspace over $\mathcal{LM}$, which can be identified with the 1-form bundle $\Omega^1(\mathcal{LM})$. Superstring states in Schrödinger representation are super-functionals on $\Omega^1(\mathcal{LM})$ and hence section of the form bundle $\Omega(\mathcal{LM})$ over loop space.

The main technical consequence of the infinite dimensionality are the well known divergencies of certain objects, such as the Ricci-Tensor and the Laplace-Beltrami operator, which inhibit the naive implementation of quantum mechanics on $\mathcal{LM}$. But of course these are just the well known infinities that arise, when working in the Heisenberg (CFT) instead of in the Schrödinger picture, from operator ordering effects, and which should be removed by imposing normal ordering. Since the choice of Schrödinger or Heisenberg picture is just one of language, the same normal ordering (now expressed in terms of functional operators instead of Fock space operators) takes care of infinities in loop space. We will therefore not have much more to say about this issue here. The main result of this section are various (deformed) representations of the super-Virasoro algebra on loop space (corresponding to different spacetime backgrounds), and will be derived in their classical (Poisson-bracket) form without considering normal ordering effects.

A mathematical discussion of aspects of loop space can for instance be found in [16, 17]. A rigorous treatment of some of the objects discussed below is also given in [18].

2.1 Definitions

Let $(\mathcal{M}, g)$ be a pseudo-riemannian manifold, the target space, with metric g, and let $\mathcal{LM}$ be its loop space consisting of smooth embeddings of the circle into $\mathcal{M}$:

$$\mathcal{LM} := C^\infty(S^1, \mathcal{M}) \,. \tag{2.1}$$

The tangent space $T_X\mathcal{LM}$ of $\mathcal{LM}$ at a loop $X : S^1 \to \mathcal{M}$ is the space of vector fields along that loop. The metric on $\mathcal{M}$ induces a metric on $T_X\mathcal{LM}$: Let $g(p) = g_{\mu\nu}(p)\, dx^\mu \otimes dx^\nu$ be the metric tensor on $\mathcal{M}$. Then we choose for the metric on $\mathcal{LM}$ at a point X the mapping

$$T_X\mathcal{LM} \times T_X\mathcal{LM} \to \mathbb{R}$$

$$
\begin{aligned}
(U,V) \mapsto U\cdot V &= \int d\sigma\; g(X(\sigma))(U(\sigma)\,,V(\sigma)) \\
&= \int d\sigma\; g_{\mu\nu}(X(\sigma))\,U^\mu(\sigma)\,V^\nu(\sigma)\;. \qquad (2.2)
\end{aligned}
$$

(For the intended applications $T\mathcal{L}\mathcal{M}$ is actually too small, since there will be need to deal with distributional vector fields on loop space. Therefore one really considers $\bar{T}\mathcal{L}\mathcal{M}$, the completion of $T\mathcal{L}\mathcal{M}$ at each point X with respect to the norm induced by the inner product (2.2).) For brevity, whenever we refer to "loop space" in the following, we mean $\mathcal{L}\mathcal{M}$ equipped with the metric (2.2).

To abbreviate the notation, let us introduce formal multi-indices (μ,σ) and write equivalently

$$
U^\mu(\sigma) := U^{(\mu,\sigma)} \qquad (2.3)
$$

for a vector $U \in T_X\mathcal{L}\mathcal{M}$, and similarly for higher-rank tensors on loop space.

Extending the usual index notation to the infinite-dimensional setting in the obvious way, we also write:

$$
\int U^\mu(\sigma)\,V_\mu(\sigma) := U^{(\mu,\sigma)}V_{(\mu,\sigma)}\,. \qquad (2.4)
$$

For this to make sense we need to know how to "shift" the continuous index σ. Because of

$$
\int d\sigma\; g_{\mu\nu}(X(\sigma))\,U^\mu(\sigma)\,V^\nu(\sigma) = \int d\sigma\; d\sigma'\;\delta\big(\sigma,\sigma'\big)\,g_{\mu\nu}(X(\sigma))\,U^\mu(\sigma)\,V^\nu\big(\sigma'\big)
$$

it makes sense to write the metric tensor on loop space as

$$
G_{(\mu,\sigma)(\nu,\sigma')}(X) := g_{\mu\nu}(X(\sigma))\,\delta\big(\sigma,\sigma'\big)\;. \qquad (2.5)
$$

Therefore

$$
\langle U,V\rangle = U^{(\mu,\sigma)}G_{(\mu,\sigma)(\nu,\sigma')}V^{(\nu,\sigma')} \qquad (2.6)
$$

and

$$
\begin{aligned}
V_{(\mu,\sigma)} &= G_{(\mu,\sigma)(\nu,\sigma')}V^{(\nu,\sigma')} \\
&= V_\mu(\sigma)\;. \qquad (2.7)
\end{aligned}
$$

Consequently, it is natural to write

$$
\delta\big(\sigma,\sigma'\big) := \delta_\sigma^{\sigma'} = \delta_{\sigma'}^{\sigma} = \delta_{\sigma,\sigma'} = \delta^{\sigma,\sigma'}\;. \qquad (2.8)
$$

A (holonomic) basis for $T_X\mathcal{L}\mathcal{M}$ may now be written as

$$
\partial_{(\mu,\sigma)} := \frac{\delta}{\delta X^\mu(\sigma)}\,, \qquad (2.9)
$$

where the expression on the right denotes the functional derivative, so that

$$
\begin{aligned}
\partial_{(\mu,\sigma)}X^{(\nu,\sigma')} &= \delta_{(\mu,\sigma)}^{(\nu,\sigma')} \\
&= \delta_\mu^\nu\,\delta\big(\sigma,\sigma'\big)\;. \qquad (2.10)
\end{aligned}
$$

By analogy, many concepts known from finite dimensional geometry carry over to the infinite dimensional case of loop spaces. Problems arise when traces over the continuous "index" σ are taken, like for contractions of the Riemann tensor, which leads to undefined diverging expressions. It is expected that these are taken care of by the usual normal-ordering of quantum field theory.

2.2 Differential geometry on loop space

With the metric (2.5) on loop space in hand

$$G_{(\mu,\sigma)(\nu,\sigma')}(X) = g_{\mu\nu}(X(\sigma))\,\delta_{\sigma,\sigma'} \tag{2.11}$$

the usual objects of differential geometry can be derived for loop space. Simple calculations yield the Levi-Civita connection as well as the Riemann curvature, which will be frequently needed later on. The exterior algebra over loop space is introduced and the exterior derivative and its adjoint, which play the central role in the construction of the super-Virasoro algebra in section 3.1 (p. 11), are constructed in terms of operators on the exterior bundle. Furthermore isometries on loop space are considered, both the one coming from reparameterization of loops as well as those induced from target space. The former leads to the reparameterization constraint on strings, while the latter is crucial for the hamiltonian evolution on loop space [4].

2.2.1 Basic geometric data.

The inverse metric is obviously

$$G^{(\mu,\sigma)(\nu,\sigma')}(X) = g^{\mu\nu}(X(\sigma))\,\delta\big(\sigma,\sigma'\big)\ . \tag{2.12}$$

A vielbein field $\mathbf{e}^a = e^a{}_\mu \mathbf{d}x^\mu$ on $\mathcal{M}$ gives rise to a vielbein field $\mathbf{E}^{(a,\sigma)}$ on loop space:

$$E^{(a,\sigma)}{}_{(\mu,\sigma')}(X) := e^a{}_\mu(X(\sigma))\,\delta^\sigma{}_{\sigma'} \tag{2.13}$$

which satisfies

$$\begin{aligned} E^{(a,\sigma)}{}_{(\mu,\sigma'')}E^{(b,\sigma)(\mu,\sigma'')} &= \eta^{ab}\delta^{\sigma,\sigma'} \\ &:= \eta^{(a,\sigma)(b,\sigma')}\,. \end{aligned} \tag{2.14}$$

For the Levi-Civita connection one finds:

$$\begin{aligned} &\Gamma_{(\mu\sigma)(\alpha\sigma')(\beta\sigma'')}(X) \\ &= \frac{1}{2}\left(\frac{\delta}{\delta X^\mu(\sigma)}G_{(\alpha,\sigma')(\beta,\sigma'')}(X) + \frac{\delta}{\delta X^\beta(\sigma'')}G_{(\mu,\sigma)(\alpha,\sigma')}(X) - \frac{\delta}{\delta X^\alpha(\sigma')}G_{(\beta,\sigma'')(\mu,\sigma)}(X)\right) \\ &= \frac{1}{2}\big((\partial_\mu G_{\alpha\beta})\big(X(\sigma')\big)\,\delta\big(\sigma,\sigma'\big)\,\delta\big(\sigma',\sigma''\big) + (\partial_\beta G_{\mu\alpha})(X(\sigma))\,\delta\big(\sigma'',\sigma\big)\,\delta\big(\sigma,\sigma'\big)\big) - \\ &\quad - \frac{1}{2}(\partial_\alpha G_{\beta\mu})\big(X(\sigma'')\big)\,\delta\big(\sigma',\sigma''\big)\,\delta\big(\sigma',\sigma\big) \\ &= \Gamma_{\mu\alpha\beta}(X(\sigma))\ \delta\big(\sigma,\sigma'\big)\,\delta\big(\sigma',\sigma''\big)\ , \end{aligned} \tag{2.15}$$

and hence

$$\Gamma_{(\mu,\sigma)}{}^{(\alpha,\sigma')}{}_{(\beta,\sigma'')}(X) = \Gamma_\mu{}^\alpha{}_\beta(X(\sigma))\ \ \delta\big(\sigma,\sigma'\big)\,\delta\big(\sigma',\sigma''\big)\ . \tag{2.16}$$

The respective connection in an orthonormal basis is

$$\begin{aligned}\omega_{(\mu,\sigma)}{}^{(a\sigma')}{}_{(b,\sigma'')}(X) &= E^{(a,\sigma')}{}_{(\alpha,\rho)}(X)\left(\delta^{(\alpha,\rho)}_{(\beta,\rho')}\partial_{(\mu,\sigma)} + \Gamma_{(\mu,\sigma)}{}^{(\alpha,\rho)}{}_{(\beta,\rho')}(X)\right)E^{(\beta,\rho')}{}_{(b,\sigma')}(X)\\ &= \omega_\mu{}^a{}_b(X(\sigma))\,\delta\big(\sigma,\sigma'\big)\,\delta\big(\sigma',\sigma''\big)\ .\end{aligned} \tag{2.17}$$

From (2.16) the Riemann tensor on loop space is obtained as

$$\begin{aligned}&R_{(\mu,\sigma_1)(\nu,\sigma_2)}{}^{(\alpha,\sigma_3)}{}_{(\beta,\sigma_4)}(X) =\\ &\quad= 2\frac{\delta}{\delta X^{[(\mu,\sigma_1)}}\Gamma_{(\nu,\sigma_2)]}{}^{(\alpha,\sigma_3)}{}_{(\beta,\sigma_4)} + 2\Gamma_{[(\mu,\sigma_1)}{}^{(\alpha,\sigma_3)}{}_{|(X,\sigma_5)|}\Gamma_{(\nu,\sigma_2)]}{}^{(X,\sigma_5)}{}_{(\beta,\sigma_4)}\\ &\quad= R_{\mu\nu}{}^\alpha{}_\beta(X(\sigma_1))\ \ \delta(\sigma_1,\sigma_2)\,\delta(\sigma_2,\sigma_3)\,\delta(\sigma_3,\sigma_4)\ .\end{aligned} \tag{2.18}$$

The Ricci tensor is formally

$$\begin{aligned}R_{(\mu,\sigma)(\nu,\sigma')}(X) &= R_{(\kappa,\sigma'')(\mu,\sigma)}{}^{(\kappa,\sigma'')}{}_{(\nu,\sigma')}(X)\\ &= R_{\mu\nu}(X(\sigma))\,\delta\big(\sigma,\sigma'\big)\ \ \delta\big(\sigma'',\sigma''\big)\ ,\end{aligned} \tag{2.19}$$

which needs to be regularized. Similarly the curvature scalar is formally

$$\begin{aligned}R(X) &= R_{(\mu,\sigma)}{}^{(\mu,\sigma)}(X)\\ &= R(X(\sigma))\,\delta^\sigma_\sigma\delta\big(\sigma'',\sigma''\big)\ .\end{aligned} \tag{2.20}$$

2.2.2 Exterior and Clifford algebra over loop space

The anticommuting fields $\mathcal{E}^{\dagger(\mu,\sigma)}$, $\mathcal{E}_{(\mu,\sigma)}$, satisfying the CAR

$$\begin{aligned}\left\{\mathcal{E}^{\dagger(\mu,\sigma)},\mathcal{E}^{\dagger(\nu,\sigma')}\right\} &= 0\\ \left\{\mathcal{E}_{(\mu,\sigma)},\mathcal{E}_{(\nu,\sigma')}\right\} &= 0\\ \left\{\mathcal{E}_{(\mu,\sigma)},\mathcal{E}^{\dagger(\nu,\sigma')}\right\} &= \delta^{(\mu,\sigma)}_{(\nu,\sigma')}\,,\end{aligned} \tag{2.21}$$

are assumed to exist over loop space, in analogy with the creators and annihilators $\hat{c}^{\dagger\mu}$, $\hat{c}_\mu$ on the exterior bundle in finite dimensions as described in [4, appendix A]. (For a mathematically rigorous treatment of the continuous CAR compare [17] and references given there.) From them the Clifford fields

$$\Gamma^{(\mu,\sigma)}_{\pm} := \mathcal{E}^{\dagger(\mu,\sigma)} \pm \mathcal{E}^{(\mu,\sigma)} \tag{2.22}$$

are obtained, which satisfy

$$\begin{aligned}\left\{\Gamma^{(\mu,\sigma)}_{\pm},\Gamma^{(\nu,\sigma')}_{\pm}\right\} &= \pm 2G^{(\mu,\sigma)(\nu,\sigma')}\\ \left\{\Gamma^{(\mu,\sigma)}_{\pm},\Gamma^{(\nu,\sigma')}_{\mp}\right\} &= 0\,.\end{aligned} \tag{2.23}$$

Since the $\Gamma_\pm$ will be related to spinor fields on the string's worldsheet, we alternatively use spinor indices $A, B, \ldots \in \{1,2\} \simeq \{+,-\}$ and write

$$\left\{\Gamma_A^{(\mu,\sigma)}, \Gamma_B^{(\nu,\sigma')}\right\} = 2 s_A \delta_{AB} G^{(\mu,\sigma)(\nu,\sigma')} \,. \tag{2.24}$$

Here s_A is defined by

$$s_+ = +1\,, \qquad s_- = -1\,.. \tag{2.25}$$

The above operators will frequently be needed with respect to some orthonormal frame $E^{(a,\sigma)}$:

$$\Gamma_A^{(a,\sigma)} := E^{(a,\sigma)}{}_{(\mu,\sigma')}\Gamma_A^{(\mu,\sigma')} \,. \tag{2.26}$$

Just like in the finite dimensional case, the following derivative operators can now be defined:

The covariant derivative operator (cf. [4, A.2]) on the exterior bundle over loop space is

$$\begin{aligned}
\hat\nabla_{(\mu,\sigma)} &= \partial^c_{(\mu,\sigma)} - \Gamma_{(\mu,\sigma)}{}^{(\alpha,\sigma')}{}_{(\beta,\sigma'')}\mathcal{E}^{\dagger(\beta,\sigma'')}\mathcal{E}_{(\alpha,\sigma')} \\
&= \partial^c_{(\mu,\sigma)} - \int d\sigma'\, d\sigma'' \Gamma_\mu{}^\alpha{}_\beta(X(\sigma))\, \delta(\sigma,\sigma')\, \delta(\sigma',\sigma'')\, \mathcal{E}^{\dagger\beta}(\sigma'')\, \mathcal{E}_\alpha(\sigma') \\
&= \partial^c_{(\mu,\sigma)} - \Gamma_\mu{}^\alpha{}_\beta(X(\sigma))\, \mathcal{E}^{\dagger\beta}(\sigma)\, \mathcal{E}_\alpha(\sigma)
\end{aligned} \tag{2.27}$$

or alternatively

$$\hat\nabla_{(\mu,\sigma)} = \partial_{(\mu,\sigma)} - \omega_\mu{}^a{}_b(X(\sigma))\, \mathcal{E}^{\dagger b}(\sigma)\, \mathcal{E}_a(\sigma) \,. \tag{2.28}$$

One should note well the difference between the functional derivative $\partial^c_{(\mu,\sigma)}$ which commutes with the coordinate frame forms ($[\partial^c_{(\mu,\sigma)}\mathcal{E}^{\dagger\nu}] = 0$) and the functional derivative $\partial_{(\mu,\sigma)}$ which instead commutes with the ONB frame forms ($[\partial_{(\mu,\sigma)}\mathcal{E}^{\dagger a}] = 0$). See [4, (A.29)] for more details.

In terms of these operators the exterior derivative and coderivative on loop space read, respectively (A.39)

$$\begin{aligned}
\mathbf{d} &= \mathcal{E}^{\dagger(\mu,\sigma)}\partial^c_{(\mu,\sigma)} \\
&= \mathcal{E}^{\dagger(\mu,\sigma)}\hat\nabla_{(\mu,\sigma)} \\
\mathbf{d}^\dagger &= -\mathcal{E}^{(\mu,\sigma)}\hat\nabla_{(\mu,\sigma)} \,.
\end{aligned} \tag{2.29}$$

We will furthermore need the form number operator

$$\mathcal{N} = \mathcal{E}^{\dagger(\mu,\sigma)}\mathcal{E}_{(\mu,\sigma)} \tag{2.30}$$

as well as its *modes*: Let $\xi : S^1 \to \mathbb{C}$ be a smooth function then

$$\mathcal{N}_\xi := \int d\sigma\, \xi(\sigma)\, \mathcal{E}^{\dagger\mu}(\sigma)\, \mathcal{E}_\mu(\sigma) \tag{2.31}$$

is the ξ-mode of the form number operator. Commuting it with the exterior derivative yields the modes of that operator:

$$\begin{aligned}\mathbf{d}_\xi &:= [\mathcal{N}_\xi, \mathbf{d}] \\ &= \int d\sigma\, \xi(\sigma)\; \mathcal{E}^{\dagger\mu}(\sigma)\, \hat{\nabla}_\mu(\sigma) \\ \mathbf{d}^\dagger{}_\xi &:= -\left[\mathcal{N}_\xi, \mathbf{d}^\dagger\right] \\ &= -\int d\sigma\, \xi(\sigma)\; \mathcal{E}^{\mu}(\sigma)\, \hat{\nabla}_\mu(\sigma)\ . \end{aligned} \tag{2.32}$$

These modes will play a crucial role in section 3 (p. 11).

2.2.3 Isometries

Regardless of the symmetries of the metric g on $\mathcal{M}$, loop space $(\mathcal{LM}, G)$ has an isometry generated by the reparameterization flow vector field K, which is defined by:[2]

$$K^{(\mu,\sigma)}(X) = T\, X'^{\mu}(\sigma)\ . \tag{2.33}$$

(Here T is just a constant which we include for later convenience.) The flow generated by this vector field obviously rotates the loops around. Since the metric (2.11) is "diagonal" in the parameter σ, this leaves the geometry of loop space invariant, and the vector field K satisfies Killing's equation

$$G_{(\nu,\sigma')(X,\sigma'')}\nabla_{(\mu,\sigma)}K^{(X,\sigma'')} + G_{(\mu,\sigma)(X,\sigma'')}\nabla_{(\nu,\sigma')}K^{(X,\sigma'')} = 0\,, \tag{2.34}$$

as is readily checked.

The Lie-derivative along K is (see [4, section A.4])

$$\begin{aligned}\mathcal{L}_K &= \left\{\mathcal{E}^{\dagger(\mu,\sigma)}\partial^c_{(\mu,\sigma)}, \mathcal{E}_{(\nu,\sigma')}X'^{(\nu,\sigma')}\right\} \\ &= X'^{(\mu,\sigma)}\partial^c_{(\mu,\sigma)} + \mathcal{E}^{\dagger(\mu,\sigma)}\mathcal{E}_{(\nu,\sigma')}\delta'_{\sigma',\sigma} \\ &= X'^{(\mu,\sigma)}\partial^c_{(\mu,\sigma)} + \mathcal{E}^{\dagger\prime(\mu,\sigma)}\mathcal{E}_{(\mu,\sigma)}\,. \end{aligned} \tag{2.35}$$

This operator will be seen to be an essential ingredient of the super-Virasoro algebra in section 3 (p. 11).

Apart from the generic isometry (2.33), every symmetry of the target space manifold $\mathcal{M}$ gives rise to a family of symmetries on $\mathcal{LM}$: Let v be any Killing vector on target space,

$$\nabla_{(\mu}v_{\nu)} = 0\,, \tag{2.36}$$

then every vector V on loop space of the form

$$V_\xi(X) = V_\xi^{(\mu,\sigma)}(X)\,\partial_{(\mu,\sigma)} \;:=\; v^\mu(X(\sigma))\,\xi^\sigma\partial_{(\mu,\sigma)}\,, \tag{2.37}$$

[2] Here and in the following a prime indicates the derivative with respect to the loop parameter σ: $X'(\sigma) = \partial_\sigma X(\sigma)$.

where $\xi^\sigma = \xi(\sigma)$ is some differentiable function $S^1 \to \mathbb{C}$, is a Killing vector on loop space. For the commutators one finds

$$\begin{aligned} [V_{\xi_1}, V_{\xi_2}] &= 0 \\ [V_\xi, K] &= V_{\xi'} \,. \end{aligned} \tag{2.38}$$

The reparameterization Killing vector K will be used to deform the exterior derivative on loop space as discussed in [4, section 2.1.1], and a target space induced Killing vector V_ξ will serve as a generator of parameter evolution on loop space along the lines of [4, section 2.2]. There it was found in equation (88) that the condition

$$[K, V_\xi] = 0 \tag{2.39}$$

needs to be satisfied for this to work. Due to (2.38) this means that one needs to choose $\xi = \text{const}$, i.e. use the integral lines of $V_{\xi=1}$ as the "time"-parameter on loop space. This is only natural: It means that every point on the loop is evolved equally along the Killing vector field v on target space.

3. Superconformal generators for various backgrounds

We now use the loop space technology to show that the loop space exterior derivative deformed by the reparameterization Killing vector K gives rise to the superconformal algebra which describes string propagation in purely gravitational backgrounds. General deformations of this algebra are introduced and applying these we find representations of the superconformal algebra that correspond to all the massless NS and NS-NS background fields.

(Parts of this construction were already indicated in [4], but there only the 0-modes of the generators and only a subset of massless bosonic background fields was considered, without spelling out the full nature of the necessary constructions on loop space.)

3.1 Purely gravitational background

In this subsection it is described how to obtain a representation of the classical super-Virasoro algebra on loop space. For a trivial background the construction itself is relatively trivial and, possibly in different notation, well known. The point that shall be emphasized here is that the identification of super-Virasoro generators with modes of the deformed exterior(co-)derivative on loop space allows a convenient treatment of curved backgrounds as well as more general non-trivial background fields.

As was discussed in [4, section 2.1.1], (which is based on [2, 3]), one may obtain from the exterior derivative and its adjoint on a manifold the generators of a global $D = 2$, $N = 1$ superalgebra by deforming with a Killing vector. The generic Killing vector field on loop space is the reparameterization generator (2.33). Using this to deform the exterior derivative and its adjoint as in [4, equation (19)] yields the operators

$$\begin{aligned} \mathbf{d}_K &:= \mathbf{d} + i\mathcal{E}_{(\mu,\sigma)} X'^{(\mu,\sigma)} \\ \mathbf{d}^\dagger{}_K &:= \mathbf{d}^\dagger - i\mathcal{E}^\dagger_{(\mu,\sigma)} X'^{(\mu,\sigma)} \,, \end{aligned} \tag{3.1}$$

(where for convenience we set $T = 1$ for the moment) which generate a *global* superalgebra. Before having a closer look at this algebra let us first enlarge it to a local superalgebra by considering the *modes* defined by

$$\begin{aligned} \mathbf{d}_{K,\xi} &:= [\mathcal{N}_\xi, \mathbf{d}^*_K] \\ {\mathbf{d}^\dagger}_{K,\xi^*} &:= -\left[\mathcal{N}_\xi, {\mathbf{d}^\dagger}^*_K\right], \end{aligned} \tag{3.2}$$

where $\cdot^*$ is the complex adjoint and $\mathcal{N}_\xi$ is the ξ-mode of the form number operator discussed in (2.31). They explicitly read

$$\begin{aligned} \mathbf{d}_{K,\xi} &= \int d\sigma\; \xi(\sigma) \left(\mathcal{E}^{\dagger\mu}(\sigma)\, \partial^c_\mu(\sigma) + i\mathcal{E}_\mu(\sigma)\, X'^\mu(\sigma)\right) \\ {\mathbf{d}^\dagger}_{K,\xi} &= -\int d\sigma\; \xi(\sigma) \left(\mathcal{E}^\mu(\sigma)\, \nabla_\mu(\sigma) + i\mathcal{E}^\dagger_\mu(\sigma)\, X'^\mu(\sigma)\right). \end{aligned} \tag{3.3}$$

Making use of the fact that $\mathbf{d}_{K,\xi}$ is actually independent of the background metric, it is easy to establish the algebra of these operators. We do this for the "classical" fields, ignoring normal ordering effects and the anomaly:

The anticommutator of the operators (3.2) with themselves defines the ξ-mode $\mathcal{L}_{K,\xi}$ of the Lie-derivative $\mathcal{L}_K$ along K:

$$\{\mathbf{d}_{K,\xi_1}, \mathbf{d}_{K,\xi_2}\} = 2i\mathcal{L}_{K,\xi_1\xi_2}\,, \tag{3.4}$$

where

$$\mathcal{L}_\xi = \int d\sigma\; \left(\xi(\sigma)\, X'^\mu(\sigma)\, \partial^c_\mu(\sigma) + \sqrt{\xi}\left(\sqrt{\xi}\mathcal{E}^{\dagger\mu}\right)'(\sigma)\, \mathcal{E}_\mu(\sigma)\right). \tag{3.5}$$

We say that a field $A(\sigma)$ has *reparameterization weight* w if

$$\begin{aligned} \left[\mathcal{L}_\xi, A(\sigma)\right]_\iota &= \left(\xi A' + w\xi' A\right)(\sigma) \\ \left[\mathcal{L}_{\xi_1}, A_{\xi_2}\right]_\iota &= A_{(w-1)\xi_1'\xi_2 - \xi_1\xi_2'}\,, \end{aligned} \tag{3.6}$$

where $A_\xi := \int d\sigma\; \xi A$. For the basic fields we find

$$\begin{aligned} w(X^\mu) &= 0 \\ w\left(X'^\mu\right) &= 1 \\ w\left(\partial^c_\mu\right) &= 1 \\ w\left(\Gamma^\mu_\pm\right) &= \frac{1}{2}\,. \end{aligned} \tag{3.7}$$

Because of $w(AB) = w(A) + w(B)$ it follows that $\mathbf{d}_{K,\xi}$ and ${\mathbf{d}^\dagger}_{K,\xi}$ are modes of integrals over densities of reparameterization weight $w = 3/2$. This implies in particular that

$$[\mathcal{L}_{\xi_1}, \mathbf{d}_{K,\xi_2}] = \mathbf{d}_{K,(\frac{1}{2}\xi_1'\xi_2 - \xi_1\xi_2')} \tag{3.8}$$

$$[\mathcal{L}_{K,\xi_1}, \mathcal{L}_{K,\xi_2}] = \mathcal{L}_{K,(\xi_1'\xi_2 - \xi_1\xi_2')}\,. \tag{3.9}$$

By taking the adjoint of (3.4) and (3.8) (or by doing the calculation explicitly), analogous relations are found for $\mathbf{d}^\dagger{}_{K,\xi}$:

$$\left\{\mathbf{d}^\dagger{}_{K,\xi_1}, \mathbf{d}^\dagger{}_{K,\xi_2}\right\} = 2i\mathcal{L}_{K,\xi_1\xi_2} \tag{3.10}$$

$$\left[\mathcal{L}_{K,\xi_1}, \mathbf{d}^\dagger{}_{K,\xi_2}\right] = \mathbf{d}^\dagger{}_{K,(\frac{1}{2}\xi_1'\xi_2-\xi_1\xi_2')}\,. \tag{3.11}$$

Equations (3.4), (3.8), and (3.10) give us part of the sought-after algebra. A very simple and apparently unproblematic but rather crucial step for finding the rest is to now define the *modes of the deformed Laplace-Beltrami operator* as the right hand side of

$$\left\{\mathbf{d}_{K,\xi_1}, \mathbf{d}^\dagger{}_{K,\xi_2}\right\} = \mathbf{\Delta}_{K,\xi_1\xi_2}\,. \tag{3.12}$$

For this definition to make sense one needs to check that

$$\left\{\mathbf{d}_{K,\xi_1\xi_3}, \mathbf{d}^\dagger{}_{K,\xi_2}\right\} = \left\{\mathbf{d}_{K,\xi_1}, \mathbf{d}^\dagger{}_{K,\xi_2\xi_3}\right\}\,. \tag{3.13}$$

It is easy to verify that this is indeed true for the operators as given in (3.3). However, in section 3.2 (p. 14) it is found that this condition is a rather strong constraint on the admissible perturbations of these operators, and the innocent looking equation (3.13) plays a pivotal role in the algebraic construction of superconformal field theories in the present context.

With $\mathbf{\Delta}_{K,\xi}$ consistently defined as in (3.12) all remaining brackets follow by using the Jacobi-identity:

$$\begin{aligned}
\left[\frac{1}{2}\mathbf{\Delta}_{K,\xi_1}, \mathbf{d}_{K,\xi_2}\right] &= i\mathbf{d}^\dagger{}_{K,(\frac{1}{2}\xi_1'\xi_2-\xi_1\xi_2')}\\
\left[\frac{1}{2}\mathbf{\Delta}_{K,\xi_1}, \mathbf{d}^\dagger{}_{K,\xi_2}\right] &= i\mathbf{d}_{K,(\frac{1}{2}\xi_1'\xi_2-\xi_1\xi_2')}\\
\left[\frac{1}{2}\mathbf{\Delta}_{K,\xi_1}, \frac{1}{2}\mathbf{\Delta}_{K,\xi_2}\right] &= -\,\mathcal{L}_{K,(\xi_1'\xi_2-\xi_1\xi_2')}\,.
\end{aligned} \tag{3.14}$$

In order to make the equivalence to the super-Virasoro algebra of the algebra thus obtained more manifest consider the modes of the K-deformed Dirac-Kähler operators on loop space:

$$\begin{aligned}
\mathbf{D}_{K,\pm} &:= \mathbf{d}_K \pm \mathbf{d}^\dagger{}_K\\
&= \Gamma_{\mp}^{(\mu,\sigma)}\left(\hat{\nabla}_{(\mu,\sigma)} \mp iTX'_{(\mu,\sigma)}\right)\\
\mathbf{D}_{K,\pm,\xi} &:= \mathbf{d}_{K,\xi} \pm \mathbf{d}^\dagger{}_{K,\xi}\,.
\end{aligned} \tag{3.15}$$

They are the supercharges which generate the super-Virasoro algebra in the usual chiral form

$$\begin{aligned}
\{\mathbf{D}_{K,\pm,\xi_1}, \mathbf{D}_{K,\pm,\xi_2}\} &= 4\left(\pm\frac{1}{2}\mathbf{\Delta}_{\xi_1\xi_2} + i\mathcal{L}_{\xi_1\xi_2}\right)\\
\left[\pm\frac{1}{2}\mathbf{\Delta}_{K,\xi_1} + i\mathcal{L}_{\xi_1}, \mathbf{D}_{K,\pm,\xi_2}\right] &= 2\mathbf{D}_{K,\pm,\frac{1}{2}\xi_1'\xi_2-\xi_1\xi_2'}\\
\left[\pm\frac{1}{2}\mathbf{\Delta}_{K,\xi_1} + i\mathcal{L}_{\xi_1}, \pm\frac{1}{2}\mathbf{\Delta}_{K,\xi_2} + i\mathcal{L}_{\xi_2}\right] &= 2i\left(\pm\frac{1}{2}\mathbf{\Delta}_{K,\xi_1'\xi_2-\xi_1\xi_2'} + i\mathcal{L}_{\xi_1'\xi_2-\xi_1\xi_2'}\right).
\end{aligned} \tag{3.16}$$

It is easily seen that this acquires the standard form when we set $\xi(\sigma) = e^{in\sigma}$ for $n \in \mathbb{N}$. In order to make the connection with the usual formulation more transparent consider a flat target space. If we define the operators

$$\mathcal{P}_{\pm,(\mu,\sigma)} := \frac{1}{\sqrt{2T}}\left(-i\partial_{(\mu,\sigma)} \pm T X'_{(\mu,\sigma)}\right) \tag{3.17}$$

with commutator

$$\left[\mathcal{P}_{A,(\mu,\sigma)}, \mathcal{P}_{B,(\nu,\sigma')}\right] = i s_A \delta_{AB} \eta_{\mu\nu} \delta'_{\sigma,\sigma'}\,, \qquad \text{for } g_{\mu\nu} = \eta_{\mu\nu} \tag{3.18}$$

we get, up to a constant factor, the usual modes

$$\begin{aligned}
\mathbf{D}_{K,\pm,\xi} &= \sqrt{2T} i \int d\sigma\, \xi(\sigma)\, \Gamma^{\mu}_{\mp}(\sigma)\, \mathcal{P}_{\mu,\mp}(\sigma)\\
\mathbf{D}^2_{K,\pm,\xi^2} &= \pm 2T \int d\sigma\, \left(\xi^2(\sigma)\, \mathcal{P}_{\mp}(\sigma)\cdot\mathcal{P}_{\mp}(\sigma) - \frac{i}{2}\xi(\sigma)\, (\xi\Gamma_{\mp})'(\sigma)\cdot\Gamma_{\mp}(\sigma)\right)\,.
\end{aligned} \tag{3.19}$$

3.2 Isomorphisms of the superconformal algebra

The representation of the superconformal algebra as above is manifestly of the form considered in [4, section 2.1.1]. We can therefore now study isomorphisms of the algebra along the lines of section 2.1.2 of that paper in order to obtain new SCFTs from known ones.

From [4, section 2.1.2] it follows that the general continuous isomorphism of the 0-mode sector ($\xi = 1$) of the algebra (3.16) is induced by some operator

$$\mathbf{W} = \int d\sigma\; W(\sigma)\;, \tag{3.20}$$

where W is a field on loop space of unit reparameterization weight

$$w(W) = 1\,, \tag{3.21}$$

and looks like

$$\begin{aligned}
\mathbf{d}_{K,1} \mapsto \mathbf{d}^{\mathbf{W}}_{K,1} &:= \exp(-\mathbf{W})\,\mathbf{d}_{K,1}\exp(\mathbf{W})\\
\mathbf{d}^{\dagger}{}_{K,1} \mapsto \mathbf{d}^{\dagger}{}^{\mathbf{W}}_{K,1} &:= \exp\left(\mathbf{W}^{\dagger}\right)\mathbf{d}^{\dagger}{}_{K,1}\exp\left(-\mathbf{W}^{\dagger}\right)\\
\mathbf{\Delta}_{K,1} \mapsto \mathbf{\Delta}^{\mathrm{W}}_{K,1} &:= \left\{\mathbf{d}^{\mathbf{W}}_{K,1}, \mathbf{d}^{\dagger}{}^{\mathbf{W}}_{K,1}\right\}\\
\mathcal{L}_1 \mapsto \mathcal{L}_1\,.
\end{aligned} \tag{3.22}$$

This construction immediately generalizes to the full algebra including all modes

$$\begin{aligned}
\mathbf{d}_{K,\xi} \mapsto \mathbf{d}^{\mathbf{W}}_{K,\xi} &:= \exp(-\mathbf{W})\,\mathbf{d}_{K,\xi}\exp(\mathbf{W})\\
\mathbf{d}^{\dagger}{}_{K,\xi} \mapsto \mathbf{d}^{\dagger}{}^{\mathbf{W}}_{K,\xi} &:= \exp\left(\mathbf{W}^{\dagger}\right)\mathbf{d}^{\dagger}{}_{K,\xi}\exp\left(-\mathbf{W}^{\dagger}\right)\\
\mathcal{L}_{\xi} \mapsto \mathcal{L}_{\xi}
\end{aligned} \tag{3.23}$$

if the crucial relation

$$\mathbf{\Delta}^{\mathbf{W}}_{K,\xi_1\xi_2} = \left\{\mathbf{d}^{\mathbf{W}}_{K,\xi_1}, \mathbf{d}^{\dagger}{}^{\mathbf{W}}_{K,\xi_2}\right\} \tag{3.24}$$

remains well defined, i.e. if (3.13) remains true:

$$\left\{\mathbf{d}^{\mathbf{W}}_{K,\xi_1\xi_3}, \mathbf{d}^{\dagger\mathbf{W}}_{K,\xi_2}\right\} = \left\{\mathbf{d}^{\mathbf{W}}_{K,\xi_1}, \mathbf{d}^{\dagger\mathbf{W}}_{K,\xi_2\xi_3}\right\}. \tag{3.25}$$

The form of these deformations follows from the fact that no matter which background fields are turned on, the generator (3.5) of spatial reparameterizations (at fixed worldsheet time) remains the same, because the string must be reparameterization invariant in any case. Preservation of the relation $\mathbf{d}_K^2 = i\mathcal{L}_K$, which says that $\mathbf{d}_K$ is nilpotent up to reparameterizations, then implies that $\mathbf{d}_K$ may transform under a similarity transformation as in the first line of (3.23). The rest of (3.23) then follows immediately.

Since this is an important point, at the heart of the approach presented here, we should also reformulate it in a more conventional language. Let L_m, $\bar{L}_m$, G_m, $\bar{G}_m$ be the holomorphic and antiholomorphic modes of the super Virasoro algebra. As discussed in section 3.1 (p. 11) we have

$$\begin{aligned}
\mathbf{\Delta}_{K,\xi} &\propto L_m + \bar{L}_{-m}\\
\mathcal{L}_{K,\xi} &\propto L_m - \bar{L}_{-m}\\
\mathbf{d}_{K,\xi} &\propto iG_m + \bar{G}_{-m}\\
\mathbf{d}^{\dagger}{}_{K,\xi} &\propto -iG_m + \bar{G}_{-m}\,,
\end{aligned} \tag{3.26}$$

with $\xi(\sigma) = e^{-im\sigma}$, as well as

$$\mathbf{W} \propto \sum_n W_n \bar{W}_n\,, \tag{3.27}$$

where W_m and $\bar{W}_m$ are the modes of the holomorphic and antiholomorphic parts of $\mathbf{W}$, which have weight h and $\bar{h}$, respectively. The goal is to find a deformation of (3.26) such that $L_m - \bar{L}_{-m}$ is preserved. Since this is the square of $\pm iG_m + \bar{G}_{-m}$ the latter may receive a similarity transformation which does not affect $L_m - \bar{L}_{-m}$ itself. Using $[L_m, W_n] = ((h-1)m - n)W_{n+m}$ and similarly for the antiholomorphic part we see that this is the case for

$$\begin{aligned}
iG_m + \bar{G}_{-m} &\to \exp\left(-\sum_n W_n\bar{W}_n\right)\left(iG_m + \bar{G}_{-m}\right)\exp\left(\sum_n W_n\bar{W}_n\right)\\
-iG_m + \bar{G}_{-m} &\to \exp\left(\sum_m \bar{W}_n^\dagger W_n^\dagger\right)\left(-iG_m + \bar{G}_{-m}\right)\exp\left(-\sum_n \bar{W}_n^\dagger W_n^\dagger\right)
\end{aligned} \tag{3.28}$$

with

$$h + \bar{h} = 1\,, \tag{3.29}$$

because then

$$L_m - \bar{L}_{-m} \to \exp\left(-\sum_n W_n\bar{W}_n\right)\left(L_m - \bar{L}_{-m}\right)\exp\left(\sum_n W_n\bar{W}_n\right) = L_m - L_{-m}\,. \tag{3.30}$$

The point of the loop-space formulation above is to clarify the nature of these deformations, which in terms of the $L_m, \bar{L}_m, G_m, \bar{G}_m$ look somewhat peculiar. In the loop space formulation it becomes manifest that we are dealing here with a generalization of the deformations first considered in [2] for supersymmetric quantum mechanics, where the supersymmetry generators are the exterior derivative and coderivative and are sent by two different similarity transformations to two new nilpotent supersymmetry generators. This and the relation to the present approach to superstrings is discussed in detail in [4, section 2.1].

Every operator $\mathbf{W}$ which satisfies (3.21) and (3.24) hence induces a classical algebra isomorphism of the superconformal algebra (3.16). (Quantum corrections to these algebras can be computed and elimination of quantum anomalies will give background equations of motion, but this shall not be our concern here.) Finding such $\mathbf{W}$ is therefore a task analogous to finding superconformal lagrangians in 2 dimensions.

However, two different $\mathbf{W}$ need not induce two different isomorphisms. In particular, *anti-hermitean* $\mathbf{W}^\dagger = -\mathbf{W}$ induce *pure gauge* transformations in the sense that all algebra elements are transformed by the *same* unitary similarity transformation

$$\mathbf{X} \mapsto e^{-\mathbf{W}}\mathbf{X}e^{\mathbf{W}} \qquad \text{for } \mathbf{X} \in \left\{\mathbf{d}_{K,\xi}, \mathbf{d}^\dagger{}_{K,\xi}, \mathbf{\Delta}_{K,\xi}, \mathcal{L}_\xi\right\} \text{ and } \mathbf{W}^\dagger = -\mathbf{W}\,. \tag{3.31}$$

Examples for such unitary transformations are given in section 3.3.4 (p. 19) and section 4.2 (p. 28). They are related to background gauge transformations as well as to string dualities. For a detailed discussion of the role of such automorphism in the general framework of string duality symmetries see [14, section 7].

In the next subsections deformations of the above form are studied in general terms and by way of specific examples.

3.3 NS-NS backgrounds

We start by deriving superconformal deformations corresponding to background fields in the NS-NS sector of the closed Type II string. Since the conformal weight of an NS-NS vertex comes from a *single* Wick contraction with the superconformal generators, while that of a spin field, which enters R-sector vertices, comes from a *double* Wick contraction, the deformation theory of NS-NS backgrounds is much more transparent than that of NS-R or NS-NS sectors, as will be made clear in the following.

3.3.1 Gravitational background by algebra isomorphism

First we reconsider the purely gravitational background from the point of view that its superconformal algebra derives from the superconformal algebra for *flat* cartesian target space by a deformation of the form (3.23). For the point particle limit this was discussed in equations (38)-(42) of [4] and the generalization to loop space is straightforward: Denote by

$$\mathbf{d}^\eta_{K,1} := \mathcal{E}^{\dagger(\mu,\sigma)}\partial_{(\mu,\sigma)} + i\mathcal{E}_{(\mu,\sigma)}X'^{(\mu,\sigma)} \tag{3.32}$$

the K-deformed exterior derivative on *flat* loop space and define the deformation operator by

$$\begin{aligned}\mathbf{W} &= \mathcal{E}^\dagger\cdot(\ln E)\cdot\mathcal{E} \\ &= \int d\sigma\; \mathcal{E}^\dagger(\sigma)\cdot(\ln e(X(\sigma)))\cdot\mathcal{E}\,,\end{aligned} \tag{3.33}$$

where $\ln E$ is the logarithm of a vielbein (2.13) on loop space, regarded as a matrix. This $\mathbf{W}$ is constructed so as to satisfy

$$e^{\mathbf{W}}\mathcal{E}^{\dagger a}(\sigma)\,e^{-\mathbf{W}} = \sum_\nu e^a{}_\nu \mathcal{E}^{\dagger(b=\nu)}\,, \tag{3.34}$$

which yields

$$\begin{aligned} e^{\mathbf{W}}\mathcal{E}^{\dagger\mu}(\sigma)\,e^{-\mathbf{W}} &= e^{\mathbf{W}}e^\mu{}_a\mathcal{E}^{\dagger a}(\sigma)\,e^{-\mathbf{W}} \\ &= e^\mu{}_a e^a{}_\nu \mathcal{E}^{\dagger(b=\nu)} \\ &= \mathcal{E}^{\dagger(b=\mu)}\,. \end{aligned} \tag{3.35}$$

Since $e^{\mathbf{W}}$ interchanges between two different vielbein fields which define two different metric tensors the index structure becomes a little awkward in the above equations. Since we won't need these transformations for the further developments we don't bother to introduce special notation to deal with this issue more cleanly. The point here is just to indicate that a $e^{\mathbf{W}}$ with the above properties does exist. It replaces all p-forms with respect to E by p-forms with respect to the flat metric. One can easily convince oneself that hence the operator $\mathbf{d}_K$ associated with the metric $G = E^2$ is related to the operator $\mathbf{d}^\eta_K$ for flat space by

$$\mathbf{d}_{K,\xi} = e^{-\mathbf{W}}\mathbf{d}^\eta_{K,\xi}e^{\mathbf{W}}\,. \tag{3.36}$$

Therefore, indeed, $\mathbf{W}$ of (3.33) induces a gravitational field on the target space.

As was discussed on [4, p. 10] we need to require $\det e = 1$, and hence

$$\mathrm{tr}\ln e = 0 \tag{3.37}$$

in order that $\mathbf{d}^{\dagger\mathbf{W}}_{K,\xi} = (\mathbf{d}_{K,\xi^*})^\dagger$. This is just a condition on the nature of the coordinate system with respect to which the metric is constructed in our framework. As an abstract operator $\mathbf{d}_{K,\xi}$ is of course *independent* of any metric, its representation in terms of the operators $X^{(\mu,\sigma)}$, $\partial_{(\mu,\sigma)}$, $\mathcal{E}^{\dagger\mu}$, $\mathcal{E}^\mu$ is not, which is what the above is all about.

Note furthermore, that

$$\mathbf{W}^\dagger = \pm\mathbf{W} \Leftrightarrow (\ln e)^{\mathrm{T}} = \pm\ln e\,. \tag{3.38}$$

According to (3.31) this implies that the antisymmetric part of $\ln e$ generates a pure gauge transformation and *only the (traceless) symmetric part* of $\ln e$ is responsible for a perturbation of the gravitational background. A little reflection shows that the gauge transformation induced by antisymmetric $\ln e$ is a rotation of the vielbein frame. For further discussion of this point see [19, p. 58].

3.3.2 B-field background

As in [4, section 2.1.3] we now consider the Kalb-Ramond B-field 2-form

$$B = \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu \tag{3.39}$$

on target space with field strength $H = dB$. This induces on loop space the 2-form

$$B_{(\mu,\sigma)(\nu,\sigma')}(X) = B_{\mu\nu}(X(\sigma))\,\delta_{\sigma,\sigma'}\,. \tag{3.40}$$

We will study the deformation operator

$$\begin{aligned}
\mathbf{W}^{(B)}(X) &:= \frac{1}{2}B_{(\mu,\sigma)(\nu,\sigma')}(X)\,\mathcal{E}^{\dagger(\mu,\sigma)}\mathcal{E}^{\dagger(\nu,\sigma')} \\
&:= \int d\sigma\;\frac{1}{2}B_{\mu\nu}(X(\sigma))\,\mathcal{E}^{\dagger\mu}(\sigma)\,\mathcal{E}^{\dagger\nu}(\sigma)
\end{aligned} \tag{3.41}$$

on loop space (which is manifestly of reparameterization weight 1) and show that the superconformal algebra that it induces is indeed that found by a canonical treatment of the usual supersymmetric σ-model with gravitational and Kalb-Ramond background.

When calculating the deformations (3.23) explicitly for $\mathbf{W}$ as in (3.41) one finds

$$\begin{aligned}
\mathbf{d}^{(B)}_{K,\xi} &:= \exp\Big(-\mathbf{W}^{(B)}\Big)\,\mathbf{d}_{K,\xi}\exp\Big(\mathbf{W}^{(B)}\Big) \\
&= \mathbf{d}_{K,\xi} + \Big[\mathbf{d}_{K,\xi},\mathbf{W}^{(b)}\Big] \\
&= \int d\sigma\;\xi\,\Big(\mathcal{E}^{\dagger\mu}\hat{\nabla}_\mu + iT\mathcal{E}_\mu X'^\mu + \frac{1}{6}H_{\alpha\beta\gamma}(X)\,\mathcal{E}^{\dagger\alpha}\mathcal{E}^{\dagger\beta}\mathcal{E}^{\dagger\gamma} - iT\mathcal{E}^{\dagger\mu}B_{\mu\nu}(X)\,X'^\nu\Big) \\
\mathbf{d}^{\dagger(B)}_{K,\xi} &= \exp\Big(\mathbf{W}^{\dagger(B)}\Big)\,\mathbf{d}^{\dagger}{}_K\exp\Big(-\mathbf{W}^{\dagger(B)}\Big) \\
&= -\int d\sigma\;\xi(\sigma)\,\Big(\mathcal{E}^{\mu}\hat{\nabla}_\mu + iT\mathcal{E}^{\dagger}_\mu X'^\mu + \frac{1}{6}H_{\alpha\beta\gamma}(X)\,\mathcal{E}^{\alpha}\mathcal{E}^{\beta}\mathcal{E}^{\gamma} - iT\mathcal{E}^{\mu}B_{\mu\nu}(X)\,X'^\nu\Big)
\end{aligned} \tag{3.42}$$

This is essentially [4, equation (72)], with the only difference that here we have mode functions ξ and an explicit realization of the deformation Killing vector.

In order to check that the above is a valid isomorphim condition (3.25) must be calculated. Concentrating on the potentially problematic terms one finds

$$\begin{aligned}
\Big\{\mathbf{d}^{(B)}_{K,\xi_1},\mathbf{d}^{\dagger(B)}_{K,\xi_2}\Big\} &= \int d\sigma\;\xi_1\xi_2(\cdots) + \\
&\quad + \int d\sigma d\sigma'\xi_1(\sigma)\,\xi_2(\sigma')\,i\Big(\mathcal{E}^{\dagger}_\mu(\sigma') - \mathcal{E}^{\nu}(\sigma')\,B_{\nu\mu}(X(\sigma'))\Big)\mathcal{E}^{\dagger\mu}(\sigma)\,\delta'(\sigma',\sigma) + \\
&\quad + \int d\sigma\,d\sigma'\xi_1(\sigma)\,\xi_2(\sigma')\,i\,\Big(\mathcal{E}_\mu(\sigma) - \mathcal{E}^{\dagger\nu}(\sigma)\,B_{\nu\mu}(X(\sigma))\Big)\,\mathcal{E}^{\mu}(\sigma')\,\delta'(\sigma,\sigma') \\
&= \int d\sigma\;\xi_1\xi_2(\cdots) - i\int d\sigma\;\;\Big(\xi_1'\xi_2B_{\nu\mu}\mathcal{E}^{\nu}\mathcal{E}^{\dagger\mu} + \xi_1\xi_2'B_{\nu\mu}\mathcal{E}^{\dagger\nu}\mathcal{E}^{\mu}\Big) \\
&= \int d\sigma\;\xi_1\xi_2(\cdots)\,.
\end{aligned} \tag{3.43}$$

This expression therefore manifestly satisfies (3.25).

With hindsight this is no surprise, because (3.42) are precisely the superconformal generators in functional form as found by canonical analysis of the non-linear supersymmetric σ-model

$$S = \frac{T}{2}\int d^2\xi d^2\theta\;\,(G_{\mu\nu} + B_{\mu\nu})\,D_+\mathbf{X}^\mu D_-\mathbf{X}^\nu\,, \tag{3.44}$$

where $\mathbf{X}^\mu$ are worldsheet superfields

$$\mathbf{X}^\mu(\xi,\theta_+,\theta_-) := X^\mu(\xi) + i\theta_+\psi_+^\mu(\xi) - i\theta_-\psi_-^\mu(\xi) + i\theta_+\theta_- F^\mu(\xi)$$

and $D_\pm := \partial_{\theta_\pm} - i\theta_\pm\partial_\pm$ with $\partial_\pm := \partial_0 \pm \partial_1$ are the superderivaties. The calculation can be found in [20, section 2]. (In order to compare the final result, [20, equations (32),(33)], with our (3.42) note that our fermions $\Gamma_\pm$ are related to the fermions $\psi_\pm$ of [20] by $\Gamma_\pm = (i^{(1\mp 1)/2}\sqrt{2T})\psi_\pm$.)

3.3.3 Dilaton background

The deformation operator in (3.33) which induces the gravitational background was of the form $\mathbf{W} = \mathcal{E}^\dagger \cdot M \cdot \mathcal{E}$ with M a traceless symmetric matrix. If instead we consider a deformation of the same form but for pure trace we get

$$\mathbf{W}^{(D)} = -\frac{1}{2}\int d\sigma\ \Phi(X)\,\mathcal{E}^{\dagger\mu}\mathcal{E}_\mu\,, \tag{3.45}$$

for some real scalar field Φ on target space. This should therefore induce a dilaton background. The associated superconformal generators are (we suppress the σ dependence and the mode functions ξ from now on)

$$\begin{aligned}
\exp\left(-\mathbf{W}^{(D)}\right)\mathbf{d}_K \exp\left(\mathbf{W}^{(D)}\right) &= e^{\Phi/2}\mathcal{E}^{\dagger\mu}\left(\hat\nabla_\mu - \frac{1}{2}(\partial_\mu\Phi)\mathcal{E}^{\dagger\nu}\mathcal{E}_\nu\right) + iTe^{-\Phi/2}X'^\mu\mathcal{E}_\mu \\
\exp\left(\mathbf{W}^{(D)}\right)\mathbf{d}^\dagger{}_K \exp\left(-\mathbf{W}^{(D)}\right) &= -e^{\Phi/2}\mathcal{E}^{\mu}\left(\hat\nabla_\mu + \frac{1}{2}(\partial_\mu\Phi)\mathcal{E}^{\dagger\nu}\mathcal{E}_\nu\right) - iTe^{-\Phi/2}X'^\mu\mathcal{E}^\dagger_\mu
\end{aligned} \tag{3.46}$$

It is readily seen that for this deformation equation (3.25) is satisfied, so that these operators indeed generate a superconformal algebra.

The corresponding Dirac operators are

$$\mathbf{d}_K^{(\Phi)} \pm \mathbf{d}^{\dagger(\Phi)}_K = \Gamma^\mu_\mp\left(e^{\Phi/2}\hat\nabla_\mu \mp iTe^{-\Phi/2}G_{\mu\nu}X'^\mu\right) \mp e^{\Phi/2}\Gamma^\mu_\pm(\partial_\mu\Phi)\mathcal{E}^{\dagger\nu}\mathcal{E}_\nu\,. \tag{3.47}$$

Comparison of the superpartners of $\Gamma_{\pm,\mu}$

$$\mp\frac{1}{2}\left\{\mathbf{d}_K^{(\Phi)} \pm \mathbf{d}^{\dagger(\Phi)}_K, \Gamma_{\mp,\mu}\right\} = e^{\Phi/2}\partial_\mu \mp iTe^{-\Phi/2}G_{\mu\nu}X'^\mu + \text{fermionic terms} \tag{3.48}$$

with equation (A.10) in appendix section A (p. 37) shows that this has the form expected for the dilaton coupling of a D-string.

3.3.4 Gauge field background

A gauge field background $A = A_\mu dx^\mu$ should express itself via $B \to B - \frac{1}{T}F$, where $F = dA$ (e.g. [21, section 8.7]), if we assume A to be a U(1) connection for the moment. Since the present discussion so far refers only to closed strings and since closed strings have trivial coupling to A it is to be expected that an A-field background manifests itself as a pure gauge transformation in the present context. This motivates to investigate the deformation induced by the anti-hermitean

$$\mathbf{W} = iA_{(\mu,\sigma)}(X)\,X'^{(\mu,\sigma)} = i\int d\sigma\ A_\mu(X(\sigma))\,X'^\mu(\sigma)\ . \tag{3.49}$$

The associated superconformal generators are found to be

$$\begin{aligned} \mathbf{d}_K^{(A)(B)} &= \mathbf{d}_K^{(B)} + i\mathcal{E}^{\dagger\mu} F_{\mu\nu} X'^{\nu} \\ \mathbf{d}^{\dagger}{}_K^{(A)(B)} &= \mathbf{d}^{\dagger}{}_K^{(B)} - i\mathcal{E}^{\mu} F_{\mu\nu} X'^{\nu} . \end{aligned} \tag{3.50}$$

Comparison with (3.42) shows that indeed

$$\mathbf{d}_K^{(A)(B)} = \mathbf{d}_K^{(B-\frac{1}{T}F)} , \tag{3.51}$$

so that we can identify the background induced by (3.49) with that of the NS $U(1)$ gauge field.

Since $\exp(\mathbf{W})(X)$ is nothing but the Wilson loop of A around X, it is natural to conjecture that for a general (non-abelian) gauge field background A the corresponding deformation is the Wilson loop as well:

$$\mathbf{d}_K^{(A)} = \left(\mathrm{Tr}\mathcal{P}e^{-i\int A_\mu X'^\mu}\right) \mathbf{d}_K \left(\mathrm{Tr}\,\mathcal{P}e^{+i\int A_\mu X'^\mu}\right) , \tag{3.52}$$

where $\mathcal{P}$ indicates path ordering and Tr the trace in the Lie algebra, as usual.

3.3.5 C-field background

So far we have found deformation operators for all massless NS and NS-NS background fields. One notes a close similarity between the form of these deformation operators and the form of the corresponding vertex operators (in fact, the deformation operators are related to the vertex operators in the (-1,-1) picture. This is discussed in section 3.4.2 (p. 24)): the deformation operators for G, B and Φ are bilinear in the form creation/annihilation operators on loop space, with the bilinear form (matrix) seperated into its traceless symmetric, antisymmetric and trace part.

Interestingly, though, there is one more deformation operator obtainable by such a bilinear in the form creation/annihilation operators. It is

$$\mathbf{W}^{(C)} := \frac{1}{2}\int d\sigma\; C_{\mu\nu}(X)\, \mathcal{E}^\mu \mathcal{E}^\nu , \tag{3.53}$$

i.e. the *adjoint* of (3.41). It induces the generators

$$\begin{aligned} \mathbf{d}_{K,\xi}^{(C)} &= \int d\sigma\; \xi \Big(\mathcal{E}^{\dagger\mu}\hat{\nabla}_\mu + i\mathcal{E}_\mu X'^\mu - \mathcal{E}^\nu C_\nu{}^\mu \hat{\nabla}_\mu + \frac{1}{2}\mathcal{E}^{\dagger\alpha}\mathcal{E}^\mu\mathcal{E}^\nu(\nabla_\alpha C_{\mu\nu}) - \\ &\qquad - \frac{1}{2}C_\nu{}^\mu\mathcal{E}^\nu\mathcal{E}^\alpha\mathcal{E}^\beta(\nabla_\mu C_{\alpha\beta}) + \frac{1}{2}C^\alpha{}_\beta\mathcal{E}^\beta\mathcal{E}^\mu\mathcal{E}^\nu(\nabla_\alpha C_{\mu\nu})\Big) \\ \mathbf{d}^{\dagger}{}_{K,\xi}^{(C)} &= -\int d\sigma\; \xi \Big(\mathcal{E}^{\mu}\hat{\nabla}_\mu + i\mathcal{E}^\dagger_\mu X'^\mu - \mathcal{E}^{\dagger\nu} C_\nu{}^\mu \hat{\nabla}_\mu + \frac{1}{2}\mathcal{E}^{\alpha}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}(\nabla_\alpha C_{\mu\nu}) - \\ &\qquad - \frac{1}{2}C_\nu{}^\mu\mathcal{E}^{\dagger\nu}\mathcal{E}^{\dagger\alpha}\mathcal{E}^{\dagger\beta}(\nabla_\mu C_{\alpha\beta}) + \frac{1}{2}C^\alpha{}_\beta\mathcal{E}^{\dagger\beta}\mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu}(\nabla_\alpha C_{\mu\nu})\Big) . \end{aligned} \tag{3.54}$$

Furthermore it turns out that this deformation, too, does respect (3.25): When we again concentrate only on the potentially problematic terms we see that

$$
\begin{aligned}
\left\{\mathbf{d}^{(C)}_{K,\xi_1}, \mathbf{d}^{\dagger(C)}_{K,\xi_2}\right\} &= \int d\sigma\ \xi_1\xi_2(\cdots) + \\
&+ \left\{-\int d\sigma\ \xi_1\mathcal{E}^\nu C_\nu{}^\mu \hat{\nabla}_\mu, -i\int d\sigma\ \xi_2\mathcal{E}^\dagger_\mu X'^\mu\right\} + \\
&+ \left\{\int d\sigma\ \xi_2\mathcal{E}^{\dagger\nu} C_\nu{}^\mu \hat{\nabla}_\mu, i\int d\sigma\ \xi_1\mathcal{E}_\mu X'^\mu\right\} \\
&= \int d\sigma\ \xi_1\xi_2\,(\cdots) + \\
&+ i\int d\sigma\ \left(\xi_1\xi_2'\ \mathcal{E}^\dagger_\mu C_\nu{}^\mu \mathcal{E}^\nu + \xi_1'\xi_2\ \mathcal{E}_\mu C_\nu{}^\mu \mathcal{E}^{\dagger\nu}\right) \\
&= \int d\sigma\ \xi_1\xi_2\,(\cdots)\,.
\end{aligned}
\tag{3.55}
$$

Therefore (3.54) do generate a superconformal algebra and hence define an SCFT.

What, though, is the physical interpretation of the field C on spacetime? It is apparently not the NS 2-form field, because the generators (3.54) are different from (3.42) and don't seem to be unitarily equivalent. A possible guess would therefore be that it is the *RR 2-form* C_2, but now coupled to a D-string instead of an F-string.

The description of the F-string in an RR background would involve ghosts and spin fields, which we do not discuss here. But the coupling of the D-string to the RR 2-form is very similar to the coupling of the F-string to the Kalb-Ramond 2-form and does not involve any spin fields. That's why the above deformation might allow an interpretation in terms of D-strings in RR 2-form backgrounds.

But this needs to be further examined. A hint in this direction is that under a duality transformation which changes the sign of the dilaton, the C-field is exchanged with the B-field. This is discussed in section 4.2.2 (p. 31).

3.4 Canonical deformations and vertex operators

With all NS-NS backgrounds under control (section 3.3 (p. 16)) we now turn to a more general analysis of the deformations of section 3.2 (p. 14) that puts the results of the previous subsections in perspective and shows how general backgrounds are to be handled.

3.4.1 Review of first order canonical CFT deformations

It has been noted long ago [7] that adding an integrated background vertex operator V (a worldsheet field of weight (1,1)) to the string's action to first order induces a perturbation

$$
L_m \to L_m + \int d\sigma\ e^{-im\sigma} V(\sigma) \tag{3.56}
$$

of the Virasoro generators and a similar shift occurs for the supercurrent [6].

While in [7] this is discussed in CFT language it becomes quite transparent in canonical language: From the string's worldsheet action for gravitational $G_{\mu\nu}$, Kalb-Ramond $B_{\mu\nu}$ and dilaton Φ background one finds the classical stress-energy tensor (cf. section A (p. 37))

$$\begin{aligned} T(\sigma) = & \frac{1}{2} G^{\mu\nu} \frac{1}{\sqrt{2T}} \left(e^{\Phi/2} P_\mu + T \left(e^{\Phi/2} B_{\mu\kappa} + e^{-\Phi/2} G_{\mu\kappa} \right) X'^\kappa \right) \times \\ & \times \frac{1}{\sqrt{2T}} \left(e^{\Phi/2} P_\nu + T \left(e^{\Phi/2} B_{\nu\kappa} + e^{-\Phi/2} G_{\nu\kappa} \right) X'^\kappa \right)(\sigma) \,, \end{aligned} \tag{3.57}$$

where P_μ is the canonical momentum to X^μ.

When expanded in terms of small perturbations

$$\begin{aligned} G_{\mu\nu}(X) &= \eta_{\mu\nu} + h_{\mu\nu}(X) + \cdots \\ B_{\mu\nu}(X) &= 0 + b_{\mu\nu}(X) + \cdots \\ \Phi(X) &= 0 + \phi(X) + \cdots \end{aligned} \tag{3.58}$$

of the background fields this yields

$$\begin{aligned} T \approx & \frac{1}{2} \left(\eta^{\mu\nu} - h^{\mu\nu} \right) \times \\ & \times \left(\mathcal{P}_{+\mu} + \sqrt{\frac{T}{2}} b_{\mu\kappa} X'^\kappa + \sqrt{\frac{T}{2}} h_{\mu\kappa} X'^\kappa + \frac{1}{\sqrt{8T}} \phi P_\mu - \sqrt{\frac{T}{8}} \phi \eta_{\mu\kappa} X'^\kappa \right) \left(\cdots \right)_\nu \\ = & \frac{1}{2} \eta_{\mu\nu} \mathcal{P}_+^\mu \mathcal{P}_+^\nu - \underbrace{\frac{1}{2} h_{\mu\nu} \mathcal{P}_+^\mu \mathcal{P}_-^\nu}_{:= V_G} - \underbrace{\frac{1}{2} b_{\mu\nu} \mathcal{P}_+^\mu \mathcal{P}_-^\nu}_{:= V_B} + \underbrace{\frac{1}{2} \phi \eta_{\mu\nu} \mathcal{P}_+^\mu \mathcal{P}_-^\nu}_{:= V_\Phi} + \text{higher order terms} \,, \end{aligned} \tag{3.59}$$

where we have defined

$$\mathcal{P}_\pm^\mu(\sigma) := \frac{1}{\sqrt{2T}} \left(\eta^{\mu\nu} P_\nu \pm T X'^\nu \right)(\sigma) \,. \tag{3.60}$$

It must be noted that while the objects $\mathcal{P}_\pm$, which have Poisson-bracket

$$\left\{ \mathcal{P}_\pm^\mu(\sigma) \,, \mathcal{P}_\pm^\nu(\sigma') \right\} = \mp \eta^{\mu\nu} \delta'\left(\sigma - \sigma' \right) \,, \tag{3.61}$$

generate the current algebra of the free theory, they involve, via $P_\mu = \delta S / \delta \dot{X}^\mu$, data of the perturbed background and are hence not proportional to ∂X and $\bar{\partial} X$.

Still, the first term in (3.59) is the generator of the Virasoro algebra which is associated with the U(1)-currents $\mathcal{P}_\pm$, while the following terms are the weight (1,1) vertices V_G, V_B, V_Φ (with respect to the first term) of the graviton, 2-form and dilaton, respectively.

Hence in the sense that we regard the canonical coordinates and momenta as fundamental and hence unaffected by the background perturbation, i.e.

$$\begin{aligned} X^\mu &\to X^\mu \\ P_\mu &\to P_\mu \,, \end{aligned} \tag{3.62}$$

while only the 'coupling constants' are shifted

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu} \,, \qquad \text{etc.} \tag{3.63}$$

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we can write

$$T \to T + V\,, \tag{3.64}$$

where V denotes a collection of weight (1,1) vertices in the above sense.[3]

CFT deformations of this form are called *canonical deformations* [8, 22].

The central idea of canonical first order deformations is that the (super-) Virasoro algebra

$$\begin{aligned}
\left[T(\sigma), T(\sigma')\right] &= 2iT(\sigma')\,\delta'(\sigma-\sigma') - iT'(\sigma')\,\delta(\sigma-\sigma') + A(\sigma-\sigma') \\
\left\{T_F(\sigma), T_F(\sigma')\right\} &= -\frac{1}{2\sqrt{2}}T(\sigma')\,\delta(\sigma') + B(\sigma-\sigma') \\
\left[T(\sigma), T_F(\sigma')\right] &= \frac{3i}{2}T_F(\sigma')\,\delta'(\sigma-\sigma') - iT_F'(\sigma')\,\delta(\sigma-\sigma')
\end{aligned} \tag{3.65}$$

(where A and B are the anomaly terms) together with its chiral partner, generated by $\bar{T}$ and $\bar{T}_F$, is preserved to first order under the perturbation

$$\begin{aligned}
T(\sigma) &\to T(\sigma) + \delta T(\sigma) \\
T_F(\sigma) &\to T_F(\sigma) + \delta T_F(\sigma)
\end{aligned} \tag{3.66}$$

if, in particular,

$$\begin{aligned}
\delta T(\sigma) &= \Phi(\sigma)\,\bar{\Phi}(\sigma) \\
\delta F(\sigma) &= \Phi_F(\sigma)\,\bar{\Phi}_F(\sigma)
\end{aligned} \tag{3.67}$$

with

$$\begin{aligned}
\left[T(\sigma), \Phi(\sigma')\right] &= i\Phi(\sigma')\,\delta'(\sigma-\sigma') - i\Phi'(\sigma')\,\delta(\sigma-\sigma') \\
\left[T(\sigma), \Phi_F(\sigma')\right] &= \frac{i}{2}\Phi(\sigma')\,\delta'(\sigma-\sigma') - i\Phi'(\sigma')\,\delta(\sigma-\sigma') \\
\left[T(\sigma), \bar{\Phi}(\sigma')\right] &= 0 \\
\left[T(\sigma), \bar{\Phi}_F(\sigma')\right] &= 0
\end{aligned} \tag{3.68}$$

and analogous relations for $\delta\bar{T}$ and $\delta\bar{T}_F$.

There are however also more general fields δT, δT_F of total weight 2 and 3/2, respectively, which preserve the above super-Virasoro algebra to first order [23]. But the weight $(1,1)$ part $\Phi(\sigma)\,\bar{\Phi}(\sigma)$ is special in that it corresponds directly to the vertex operator of the background which is described by the deformed worldsheet theory. Further deformation fields of weight different from (1,1) are related to *gauge* degrees of freedom of the background fields (cf. [23] and the discussion below equation (3.79)).

[3] As is discussed in [7], the issue concerning (3.62) in CFT language translates into the question whether one chooses to treat ∂X and $\bar{\partial} X$ as free fields in the perturbed theory and whether the $\partial X\,\partial X$-OPE is taken to receive a perturbation or not.

For a further discussion of perturbations of SCFTs where this issue is addressed, see [4] and in particular section 2.2.4.

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3.4.2 Canonical deformations from $\mathbf{d}_K \to e^{-\mathbf{W}}\mathbf{d}_K e^{\mathbf{W}}$.

We would like to see how the deformation theory reviewed above relates to the SCFT deformations that have been studied in section 3 (p. 11).

First recall from section 3.1 (p. 11) that the chiral bosonic fields in our notation read

$$\mathcal{P}_\pm(\sigma) := \frac{1}{\sqrt{2T}}\left(-i\frac{\delta}{\delta X} \pm TX'\right)(\sigma) \tag{3.69}$$

and that according to section 2.2.2 (p. 8) we write the worldsheet fermions ψ, $\bar\psi$ as $\Gamma_\pm$, respectively, which are normalized so that $\left\{\Gamma^\mu_\pm(\sigma), \Gamma^\nu_\pm(\sigma')\right\} = \pm 2g^{\mu\nu}(X(\sigma))\,\delta(\sigma-\sigma')$, and we frequently make use of the linear combinations

$$\begin{aligned} \mathcal{E}^{\dagger\mu} &= \frac{1}{2}\left(\Gamma^\mu_+ + \Gamma^\mu_-\right) \\ \mathcal{E}^{\mu} &= \frac{1}{2}\left(\Gamma^\mu_+ - \Gamma^\mu_-\right). \end{aligned} \tag{3.70}$$

In this notation the supercurrents for the trivial background read

$$\begin{aligned} T_F(\sigma) &= \frac{1}{\sqrt{2}}\Gamma_+(\sigma)\,\mathcal{P}_+(\sigma) = \frac{-i}{\sqrt{4T}}\left(\mathbf{d}_K(\sigma) - \mathbf{d}^\dagger{}_K(\sigma)\right) \\ \bar T_F(\sigma) &= \frac{i}{\sqrt{2}}\Gamma_-(\sigma)\,\mathcal{P}_-(\sigma) = \frac{1}{\sqrt{4T}}\left(\mathbf{d}_K(\sigma) + \mathbf{d}^\dagger{}_K(\sigma)\right), \end{aligned} \tag{3.71}$$

where the K-deformed exterior derivative and coderivative on loop space are identified as

$$\begin{aligned} \mathbf{d}_K &= \sqrt{T}\left(\bar T_F + iT_F\right) \\ \mathbf{d}^\dagger{}_K &= \sqrt{T}\left(\bar T_F - iT_F\right). \end{aligned} \tag{3.72}$$

According to section 3.2 (p. 14) a consistent deformation of the superconformal algebra generated by T_F and $\bar T_F$ is given by sending

$$\begin{aligned} \mathbf{d}_K(\sigma) &\to \mathbf{d}^{(W)}_K(\sigma) = e^{-\mathbf{W}}\mathbf{d}_K(\sigma)\,e^{\mathbf{W}} = \mathbf{d}_K(\sigma) + \underbrace{[\mathbf{d}_K(\sigma), \mathbf{W}]}_{:=\delta\mathbf{d}_K(\sigma)} + \cdots \\ \mathbf{d}^\dagger{}_K(\sigma) &\to \mathbf{d}^{\dagger(W)}_K(\sigma) = e^{\mathbf{W}^\dagger}\mathbf{d}^\dagger{}_K(\sigma)\,e^{-\mathbf{W}^\dagger} = \mathbf{d}^\dagger{}_K(\sigma) + \underbrace{\left[\mathbf{W}^\dagger, \mathbf{d}^\dagger{}_K(\sigma)\right]}_{:=\delta\mathbf{d}^\dagger{}_K(\sigma)} + \cdots \end{aligned} \tag{3.73}$$

for $\mathbf{W}$ some reparameterization invariant operator. From this one finds δT_F by using (3.71)

$$\begin{aligned} \delta T_F &= -i\left[\bar T_F, \frac{1}{2}\left(\mathbf{W} + \mathbf{W}^\dagger\right)\right] + \left[T_F, \frac{1}{2}\left(\mathbf{W} - \mathbf{W}^\dagger\right)\right] \\ \delta \bar T_F &= i\left[T_F, \frac{1}{2}\left(\mathbf{W} + \mathbf{W}^\dagger\right)\right] + \left[\bar T_F, \frac{1}{2}\left(\mathbf{W} - \mathbf{W}^\dagger\right)\right] \end{aligned} \tag{3.74}$$

which again gives δT by means of

$$\left\{T_F(\sigma), \delta T_F(\sigma')\right\} + \left\{T_F(\sigma'), \delta T_F(\sigma)\right\} = -\frac{1}{2\sqrt{2}}\delta T(\sigma)\,\delta(\sigma - \sigma')\,. \tag{3.75}$$

Before looking at special cases one should note that this necessarily implies that δT_F is of total weight 3/2 and that δT is of total weight 2. That is because $\mathbf{W}$, being reparameterization invariant, must be the integral (along the string at fixed worldsheet time) over a field of unit total weight (cf. (3.21) and (3.29)) and because supercommutation with $\mathbf{d}_K$ or $\mathbf{d}^\dagger{}_K$ increases the total weight by 1/2.

Furthermore, recall from (3.31) that the anti-hermitean part $\frac{1}{2}\left(\mathbf{W} - \mathbf{W}^\dagger\right)$ of the deformation operator $\mathbf{W}$ is responsible for *pure gauge transformations* while the hermitean part $\frac{1}{2}\left(\mathbf{W} + \mathbf{W}^\dagger\right)$ induces true modifications of the background fields. Hence for a pure gauge transformation (3.74) yields

$$\begin{aligned} \delta T_F &= \left[T_F, \mathbf{W}\right] \\ \delta \bar{T}_F &= \left[\bar{T}_F, \mathbf{W}\right] , \qquad \text{for } \mathbf{W}^\dagger = -\mathbf{W}^\dagger , \end{aligned} \tag{3.76}$$

which of course comes from the global similarity transformation (3.31)

$$\mathbf{X} \mapsto e^{-\mathbf{W}} \mathbf{X} e^{\mathbf{W}} , \qquad \mathbf{X} \in \left\{T_F, \bar{T}_F, \cdots\right\} . \tag{3.77}$$

On the other hand, for a strictly non-gauge transformation the transformation (3.74) simplifies to

$$\begin{aligned} \delta T_F &= -i\left[\bar{T}_F, \mathbf{W}\right] \\ \delta \bar{T}_F &= i\left[T_F, \mathbf{W}\right] , \qquad \text{for } \mathbf{W}^\dagger = +\mathbf{W} . \end{aligned} \tag{3.78}$$

In the cases where $\mathbf{W}$ is antihermitean *and* a $(1/2, 1/2)$ field (as is in particular the case for the gravitational $\mathbf{W}^{(G)}$ of section 3.3.1 (p. 16), the dilaton $\mathbf{W}^{(D)}$ of section 3.3.3 (p. 19) and the hermitean part of the Kalb-Ramond $\mathbf{W}^{(B)} + \mathbf{W}^{(B)\dagger}$ of section 3.3.2 (p. 17)) this, together with (3.75) implies that

$$\delta T \propto \left\{T_F, \left[\bar{T}_F, \mathbf{W}\right]\right\} \tag{3.79}$$

is indeed of weight $(1,1)$, as discussed in the theory of canonical deformations section 3.4.1 (p. 21). Furthermore, this shows explicitly that all contributions to δT which are of total weight 2 but *not* of weight (1,1) must come from the antihermitean component $\frac{1}{2}\left(\mathbf{W} - \mathbf{W}^\dagger\right)$ and hence must be associated with background gauge transformations. (This proves in full generality the respective observation in [23] concerning 2-form field deformations.)

Finally, equation (3.79) clarifies exactly how the deformation operators $\mathbf{W}$ are related to the *vertex operators* of the respective background fields, namely it shows that the hermitean part of $\mathbf{W}$ is proportional to the vertex operator in the (-1,-1) picture (i.e. the pre-image under $\left\{T_F, \left[\bar{T}_F, \cdot\right]\right\}$).

As an example, consider the deformation induced by a B-field background: according to section 3.3.2 (p. 17) a Kalb-Ramond background is induced by choosing

$$\mathbf{W} = \int d\sigma \; \frac{1}{2} B_{\mu\nu}(X(\sigma))\, \mathcal{E}^{\dagger\mu} \mathcal{E}^{\dagger\nu} \tag{3.80}$$

which, using (3.73) gives rise to

$$\delta \mathbf{d}_K(\sigma) = \left(\frac{1}{6}H_{\alpha\beta\gamma}(X)\,\mathcal{E}^{\dagger\alpha}\mathcal{E}^{\dagger\beta}\mathcal{E}^{\dagger\gamma} - iT\mathcal{E}^{\dagger\mu}B_{\mu\nu}(X)\,X'^{\nu}\right)(\sigma)$$
$$\delta \mathbf{d}^{\dagger}{}_K(\sigma) = \left(-\frac{1}{6}H_{\alpha\beta\gamma}(X)\,\mathcal{E}^{\alpha}\mathcal{E}^{\beta}\mathcal{E}^{\gamma} + iT\mathcal{E}^{\mu}B_{\mu\nu}(X)\,X'^{\nu}\right)(\sigma) \tag{3.81}$$

and hence, using (3.71), to

$$\delta T_F(\sigma) = -\frac{i}{12\sqrt{T}}H_{\alpha\beta\gamma}\left(\mathcal{E}^{\dagger\alpha}\mathcal{E}^{\dagger\beta}\mathcal{E}^{\dagger\gamma} + \mathcal{E}^{\alpha}\mathcal{E}^{\beta}\mathcal{E}^{\gamma}\right) - \frac{1}{\sqrt{8}}\Gamma^{\mu}_{+}B_{\mu\nu}\left(\mathcal{P}^{\nu}_{+} - \mathcal{P}^{\nu}_{-}\right). \tag{3.82}$$

In this special case δT_F happens to be the exact shift of T_F (there are no higher order perturbations of T_F in this background). As has been noted already in section 3.3.2 (p. 17) the same result is obtained by canonically quantizing the supersymmetric $2d$ σ-model (3.44) which describes superstrings in a Kalb-Ramond background.

By means of (3.75) the shift δT is easily found to be

$$\begin{aligned}\delta T(\sigma) = &\left(-\frac{1}{12T}\partial_\delta H_{\alpha\beta\gamma}\left(\mathcal{E}^{\delta}\mathcal{E}^{\dagger\alpha}\mathcal{E}^{\dagger\beta}\mathcal{E}^{\dagger\gamma} + \mathcal{E}^{\dagger\delta}\mathcal{E}^{\alpha}\mathcal{E}^{\beta}\mathcal{E}^{\gamma}\right) - i\frac{1}{2\sqrt{2T}}H_{\alpha\beta\gamma}\,\times\right.\\ &\left.\times\left(\mathcal{E}^{\dagger\alpha}\mathcal{E}^{\dagger\beta} + \mathcal{E}^{\alpha}\mathcal{E}^{\beta}\right)\mathcal{P}^{\gamma}_{+} + \frac{i}{\sqrt{4T}}\partial_\delta B_{\mu\nu}\left(\mathcal{P}^{\nu}_{+} - \mathcal{P}^{\nu}_{-}\right)\Gamma^{\delta}_{+}\Gamma^{\mu}_{+} + B_{\mu\nu}\mathcal{P}^{\mu}_{+}\mathcal{P}^{\nu}_{-}\right)(\sigma)\,.\end{aligned} \tag{3.83}$$

This is of total weight 2 and contains the weight (1,1) vertex operator

$$V = B_{\mu\nu}\mathcal{P}^{\mu}_{+}\mathcal{P}^{\nu}_{-} \tag{3.84}$$

of the Kalb-Ramond field (cf. [23, eqs. (52) and (53)]). That $T + \delta T$ satisfies the Virasoro algebra to first order at the level of Poisson brackets follows from the fact that it derives from a consistent deformation of the form (3.23) (as well as from the fact that it also derives from the respective σ-model lagrangian).

4. Relations between the various superconformal algebras

We have found classical deformations of the superconformal algebra associated with several massless target space background fields. The special algebraic nature of the form in which we obtain these superconformal algebras admits a convenient treatment of gauge and duality transformations among the associated background fields. This is discussed in the following.

4.1 $\mathbf{d}_K$-exact deformation operators

Deformation operators $\mathbf{W}$ which are $\mathbf{d}_K$-exact, i.e. which are of the form

$$\mathbf{W} = \left[\mathbf{d}_K, \mathbf{w}\right]_{\iota}\,, \tag{4.1}$$

(where $[\cdot,\cdot]_{\iota}$ is the supercommutator) and which furthermore satisfy

$$\mathbf{W} = \left[\mathbf{d}_{K,\xi}, \mathbf{w}_{\xi^{-1}}\right]_{\iota}\,, \qquad \forall\,\xi \tag{4.2}$$

(where $\mathbf{w}_\xi := \int d\sigma\ \xi\ w(\sigma)$) are special because for them[4]

$$[\mathbf{d}_{K,\xi}, [\mathbf{d}_K, \mathbf{w}]_\iota]_\iota = 0 \tag{4.4}$$

and hence they leave the generators of the algebra (3.16) invariant:

$$\mathbf{d}^{\mathbf{W}}_{K,\xi} = \mathbf{d}_{K,\xi}\,. \tag{4.5}$$

Two interesting choices for $\mathbf{w}$ are

$$\mathbf{w} = A_{(\mu,\sigma)}(X)\,\mathcal{E}^{\dagger(\mu,\sigma)} \tag{4.6}$$

and

$$\mathbf{w} = V^{(\mu,\sigma)}(X)\,\mathcal{E}_{(\mu,\sigma)}\,, \tag{4.7}$$

which both satisfy (4.2). They correspond to B-field gauge transformations and to diffeomorphisms, respectively:

4.1.1 B-field gauge transformations

For the choice (4.6) one gets

$$\begin{aligned}\mathbf{W} &= \left\{\mathbf{d}_K, A_{(\mu,\sigma)}\mathcal{E}^{\dagger(\mu,\sigma)}\right\}\\ &= \frac{1}{2}(dA)_{(\mu,\sigma)(\nu,\sigma')}\mathcal{E}^{\dagger(\mu,\sigma)}\mathcal{E}^{\dagger(\nu,\sigma')} + iTA_{(\mu,\sigma)}X'^{(\mu,\sigma)}\,.\end{aligned} \tag{4.8}$$

Comparison with (3.41) and (3.49) shows that this $\mathbf{W}$ induces a B-field background with $B = dA$ and a gauge field background with $F = T\,dA$. According to (3.51) these two backgrounds indeed precisely cancel.

This ties up a loose end from section 3.3.2 (p. 17): A pure gauge transformation $B \to B + dA$ of the B-field does not affect physics of the closed string and hence should manifest itself as an algebra isomorphism. Indeed, this isomorphism is that induced by $\mathbf{W} = iTA_{(\mu,\sigma)}X'^{(\mu,\sigma)}$.

4.1.2 Target space diffeomorphisms

For the choice (4.7) one gets

$$\begin{aligned}\mathbf{W} &= \left\{\mathbf{d}_K, V^{(\mu,\sigma)}\mathcal{E}_{(\mu,\sigma)}\right\}\\ &= \int d\sigma\ \left(V^\mu\partial_\mu + (\partial_\mu V^\nu)\mathcal{E}^{\dagger\mu}\mathcal{E}_\nu\right)(\sigma)\\ &= \mathcal{L}_V\,,\end{aligned} \tag{4.9}$$

[4]One way to see this is the following:

$$\begin{aligned}\left[\mathbf{d}_{K,\xi}, [\mathbf{d}_K, \mathbf{w}]_\iota\right]_\iota &= \left[\mathbf{d}_{K,\xi}, \left[\mathbf{d}_{K,\xi}, \mathbf{w}_{\xi^{-1}}\right]_\iota\right]_\iota\\ &= \left[\mathcal{L}_{K,\xi^2}, \mathbf{w}_{\xi^{-1}}\right]_\iota\\ &= \int d\sigma\ \left(\xi^2\xi^{-1}w' + \frac{1}{2}(\xi^2)'\xi^{-1}w\right)(\sigma)\\ &= \int d\sigma\ (\xi w)'\,,\end{aligned} \tag{4.3}$$

where we used that $w(\sigma)$ must be of weight 1/2 in order that $W(\sigma)$ satisfies condition (3.21).

where $\mathcal{L}_V$ is the operator inducing the Lie derivative along V on forms over loop space (cf. A.4 of [4]). According to section 3.3.1 (p. 16) the part involving $(\partial_\mu V^\nu)\mathcal{E}^{\dagger\mu}\mathcal{E}_\nu$ changes the metric field at every point of target space, while the part involving $V^\mu\partial_\mu$ translates the fields that enter in the superconformal generators. This $\mathbf{W}$ apparently induces target space diffeomorphisms.

4.2 T-duality

It is well known ([13] and references given there) that in the context of the non-commutative-geometry description of stringy spacetime physics T-duality corresponds to an inner automorphism

$$\mathcal{T} : \mathcal{A} \to e^{-\mathbf{W}}\,\mathcal{A}\,e^{\mathbf{W}} = \mathcal{A} \qquad \text{with } \mathbf{W}^\dagger = -\mathbf{W} \tag{4.10}$$

of the algebra $\mathcal{A}$ that enters the spectral triple. This has been worked out in detail for the bosonic string in [15]. In the following this construction is adapted to and rederived in the present context for the superstring and then generalized to the various backgrounds that we have found by deformations.

Following [13] we first consider T-duality along all dimensions, or equivalently, restrict attention to the field components along the directions that are T-dualized. Then we show that the *Buscher rules* (see [24] for a recent reference) for factorized T-duality (i.e. for T-duality along only a single direction) can very conveniently be derived in our framework, too.

4.2.1 Ordinary T-duality

Since T-dualizing along spacetime directions that are not characterized by commuting isometries is a little subtle (cf. [15, section 4]), assume that a background consisting of a non-trivial metric g and Kalb-Ramond field b is given together with Killing vectors ∂_{μ_n} such that

$$\begin{aligned}\partial_{\mu_n} g_{\alpha\beta} &= 0\\ \partial_{\mu_n} b_{\alpha\beta} &= 0\,.\end{aligned} \tag{4.11}$$

For convenience of notation we restrict attention in the following to the coordinates x^{μ_n}, since all other coordinates are mere spectators when T-dualizing. Furthermore we will suppress the subindex n altogether.

The inner automorphism $\mathcal{T}$ of the algebra of operators on sections of the exterior bundle over loop space is defined by its action on the canonical fields as follows:

$$\begin{aligned}\mathcal{T}(-i\partial_\mu) &= X'^\mu\\ \mathcal{T}\left(X'^\mu\right) &= -i\partial_\mu\\[1em] \mathcal{T}\left(\mathcal{E}^{\dagger a}\right) &= \mathcal{E}_a\\ \mathcal{T}(\mathcal{E}_a) &= \mathcal{E}^{\dagger a}\,.\end{aligned} \tag{4.12}$$

It is possible (see [15] and of [13, p. 47]) to express this automorphism manifestly as a similarity transformation

$$\mathcal{T}(A) = e^{-\mathbf{W}}\, A\, e^{\mathbf{W}}\,. \tag{4.13}$$

This however requires taking into account normal ordering, which would lead us too far afield in the present discussion. For our purposes it is fully sufficient to note that $\mathcal{T}$ preserves the canonical brackets

$$\begin{aligned}\left[-i\partial_{(\mu,\sigma)}, X'^{(\nu,\sigma')}\right] &= i\delta_\mu^\nu \delta'(\sigma,\sigma') \\ &= \left[\mathcal{T}(-i\partial_{(i,\sigma)}), \mathcal{T}\left(X'^{(j,\sigma')}\right)\right]\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\left\{\mathcal{E}_{(i,\sigma)}, \mathcal{E}^{\dagger(j,\sigma)}\right\} &= \delta_i^j \delta(\sigma,\sigma') \\ &= \left\{\mathcal{T}(\mathcal{E}_{(i,\sigma)}), \mathcal{T}\left(\mathcal{E}^{\dagger(j,\sigma)}\right)\right\}\end{aligned} \tag{4.15}$$

(with the other transformed brackets vanishing) and must therefore be an algebra automorphism.

Acting on the K-deformed exterior (co)derivative on loop space the transformation $\mathcal{T}$ produces (we suppress the variable σ and the mode functions ξ for convenience)

$$\begin{aligned}\mathcal{T}(\mathbf{d}_K) &= \mathcal{T}\left(\mathcal{E}^{\dagger a} E_a{}^\mu \partial_\mu + iT\mathcal{E}_a E^a{}_\mu X'^\mu\right) \\ &= i\mathcal{E}_a E_a{}^\mu X'^\mu + T\mathcal{E}^{\dagger a} E^a{}_\mu \partial_\mu \\ &= \mathcal{E}^{\dagger a} \tilde{E}_a{}^\mu \partial_\mu + iT\mathcal{E}_a \tilde{E}^a{}_\mu X'^\mu \\ \mathcal{T}\left(\mathbf{d}^\dagger{}_K\right) &= (\mathcal{T}(\mathbf{d}_K))^\dagger\,,\end{aligned} \tag{4.16}$$

where the T-dual vielbein $\tilde{E}$ is defined as

$$\tilde{E}^a{}_\mu := \frac{1}{T} E_a{}^\mu\,. \tag{4.17}$$

(This is obviously not a tensor equation but true in the special coordinates that have been chosen.) Therefore T-duality sends the deformed exterior derivative associated with the metric defined by the vielbein $E_a{}^\mu$ to that associated with the metric defined by the vielbein $\tilde{E}_a{}^\mu$. This yields the usual inversion of the spacetime radius $R \mapsto \alpha'/R$:

$$E^a{}_\mu = \delta_\mu^a \sqrt{2\pi} R \Rightarrow \tilde{E}^a{}_\mu = \delta_\mu^a \frac{1}{T}\frac{1}{\sqrt{2\pi}R} = \delta_\mu^a \sqrt{2\pi}\,\frac{\alpha'}{R}\,. \tag{4.18}$$

Furthermore it is readily checked that the bosonic and fermionic worldsheet oscillators transform as expected:

$$\begin{aligned}\mathcal{T}(\mathcal{P}_{\pm,a}) &= \mathcal{T}\left(\frac{1}{\sqrt{2T}}\left(-iE_a{}^\mu \partial_\mu \pm TE_{a\mu}X'^\mu\right)\right) \\ &= \frac{1}{\sqrt{2T}}\left(E_a{}^\mu X'^\mu \pm -iTE_{a\mu}\partial_\mu\right) \\ &= \pm\frac{1}{\sqrt{2T}}\left(-i\tilde{E}_a{}^\mu \partial_\mu \pm TE_{a\mu}X'^\mu\right) \\ &= \pm\tilde{\mathcal{P}}_{\pm,a}\end{aligned} \tag{4.19}$$

and

$$\mathcal{T}\left(\Gamma^a_\pm\right) = \pm\Gamma^a_\pm\,. \tag{4.20}$$

More generally, when the Kalb-Ramond field is included one finds

$$\begin{aligned}\mathcal{T}\left(\mathbf{d}_K^{(B)} \pm \mathbf{d}^{\dagger}{}_K^{(B)}\right) &= \mathcal{T}\left(\Gamma^a_\mp E_a{}^\mu\left(\partial_\mu \mp iT\left(G_{\mu\nu} \pm B_{\mu\nu}\right)X'^\nu\right)\right) \\ &= \mp\Gamma^a_\mp E_a{}^\mu\left(iX'^\mu \mp T\left(G_{\mu\nu} \pm B_{\mu\nu}\right)\partial_\nu\right) \\ &= \Gamma^a_\mp \tilde{E}_a{}^\mu\left(\partial_\mu \mp \left[T\left(G_{\mu\nu} \pm B_{\mu\nu}\right)\right]^{-1}X'^\nu\right)\end{aligned} \tag{4.21}$$

with

$$\tilde{E}_a{}^\mu := TE_a{}^\nu(G_{\nu\mu} \pm B_{\nu\mu})\,, \tag{4.22}$$

which reproduces the well known result ([25, equation (2.4.39)]) that the T-dual spacetime metric is given by

$$\tilde{G}^{\mu\nu} = T^2[(G \mp B)G^{-1}(G \pm B)]_{\mu\nu} \tag{4.23}$$

and that the T-dual Kalb-Ramond field is

$$\begin{aligned}\tilde{B}_{\mu\nu} &= \pm[\frac{1}{T^2}(G \pm B)^{-1} - \tilde{G}]_{\mu\nu} \\ &= \left[T^2(G \mp B)B^{-1}(G \pm B)\right]^{-1}_{\mu\nu}\,.\end{aligned} \tag{4.24}$$

It is also very easy in our framework to derive the Buscher rules for T-duality along a single direction y ("factorized duality"): Let $\mathcal{T}_y$ be the transformation (4.12) restricted to the ∂_y direction, then from

$$\begin{aligned}\mathcal{T}\left(\mathbf{d}_K^{(B)} \pm \mathbf{d}^{\dagger}{}_K^{(B)}\right) &= \Gamma^a_\mp\left(E_a{}^i\partial_i \mp iTE_a{}^\mu\left(G_{\mu i} \pm B_{\mu i}\right)X'^i\right) + \\ &\quad + \mathcal{T}\left(\Gamma^a_\mp\left(E_a{}^y\partial_y \mp iTE^\mu\left(G_{\mu y} \pm B_{\mu y}\right)X'^y\right)\right) \\ &= \Gamma^a_\mp\left(E_a{}^i\partial_i \mp iTE_a{}^\mu\left(G_{\mu i} \pm B_{\mu i}\right)X'^i\right) + \\ &\quad + \Gamma^a_\mp\left(TE_a{}^\mu\left(G_{\mu y} \pm B_{\mu y}\right)\partial_y \mp iE_a{}^y e^{\Phi/2}X'^y\right)\end{aligned} \tag{4.25}$$

one reads off the T-dual inverse vielbein

$$\begin{aligned}\tilde{E}_a{}^i &= E_a{}^i \\ \tilde{E}_a{}^y &= TE_a{}^\mu\left(G_{\mu y} \pm B_{\mu y}\right)\end{aligned} \tag{4.26}$$

whose inverse $\tilde{E}_\mu{}^a$ is easily seen to be

$$\begin{aligned}\tilde{E}_i{}^a &= E_i{}^a - \frac{G_{iy} \pm B_{iy}}{G_{yy}}E_y{}^a \\ \tilde{E}_y{}^a &= \frac{1}{TG_{yy}}E_y{}^a\,,\end{aligned} \tag{4.27}$$

which gives the T-dual metric with minimal computational effort:

$$\begin{aligned}\tilde{G}_{yy} &= \frac{1}{TG_{yy}} \\ \tilde{G}_{iy} &= \mp\frac{B_{iy}}{TG_{yy}} \\ \tilde{G}_{ij} &= G_{ij} - \frac{1}{G_{yy}}\left(G_{iy}G_{jy} - B_{iy}B_jy\right).\end{aligned} \tag{4.28}$$

Similarly the relations

$$\tilde{E}_a{}^\mu(\tilde{G}_{\mu i} \pm \tilde{B}_{\mu i}) = E_a{}^\mu(G_{\mu i} \pm B_{\mu i})$$
$$\tilde{E}_a{}^\mu(\tilde{G}_{\mu y} \pm \tilde{B}_{\mu y}) = \frac{1}{T} E_a{}^y \tag{4.29}$$

for the T-dual B-field $\tilde{B}$ are read off from (4.25). Solving them for $\tilde{B}$ is straightforward and yields

$$\tilde{B}_{ij} = B_{ij} \mp \frac{1}{G_{yy}} \left(B_{jy} G_{iy} - B_{iy} G_{iy}\right)$$
$$\tilde{B}_{iy} = \frac{1}{T G_{yy}} G_{iy} \,. \tag{4.30}$$

These are the well known *Buscher rules* for factorized T-duality (see [25, eq. (4.1.9)]).

The constant dilaton can be formally absorbed into the string tension T and is hence seen to be invariant under $\mathcal{T}_y$. This is correct in the classical limit that we are working in. It is well known (e.g. [25, eq. (4.1.10)]), that there are higher loop corrections to the T-dual dilaton. These corrections are not visible with the methods discussed here.

Using our representation for the superconformal generators in various backgrounds it is now straightfroward to include more general background fields than just G and B in the above construction:

4.2.2 T-duality for various backgrounds

When turning on all fields G, B, A, C and Φ, requiring them to be constant in the sense of (4.11) and assuming for convenience of notation that $B \cdot C = 0$ the supercurrents read according to the considerations in section 3.1–section 3.3.4

$$\mathbf{d}_K^{(\Phi)(A+B+C)} \pm \mathbf{d}^\dagger{}_K^{(\Phi)(A+B+C)} = \tag{4.31}$$
$$= \Gamma^a_\mp E_a{}^\mu \left(e^{\Phi/2}(G_\mu{}^\nu \pm C_\mu{}^\nu)\partial_\nu \mp iTe^{-\Phi/2}(G_{\mu\nu} \pm \left(B_{\mu\nu} + \frac{1}{T} F_{\mu\nu}\right) X'^\nu \right).$$

It is straightforward to apply $\mathcal{T}$ to this expression and read off the new fields. However, since the resulting expressions are not too enlightening we instead use a modification $\tilde{\mathcal{T}}$ of $\mathcal{T}$, which, too, induces an algebra isomorphism, but which produces more accessible field redefinitions. The operation $\tilde{\mathcal{T}}$ differs from $\mathcal{T}$ in that index shifts are included:

$$\tilde{\mathcal{T}}(-i\partial_\mu) := T g_{\mu\nu} X'^\nu$$
$$\tilde{\mathcal{T}}\left(X'^\mu\right) := -\frac{i}{T} g^{\mu\nu} \partial_\nu$$
$$\tilde{\mathcal{T}}\left(\mathcal{E}^{\dagger a}\right) := \mathcal{E}^a$$
$$\tilde{\mathcal{T}}(\mathcal{E}^a) := \mathcal{E}^{\dagger a} \,. \tag{4.32}$$

Due to the constancy of $g_{\mu\nu}$ this preserves the canonical brackets just as in (4.14) and hence is indeed an algebra isomorphism.

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Applying it to the supercurrents (4.31) yields

$$\tilde{\mathcal{T}}\left[\mathbf{d}_K^{(\Phi)(B+C)} \pm \mathbf{d}_K^{(\Phi)(B+C)}\right] =$$
$$= \Gamma^a_{\mp} E_a{}^\mu \left(e^{-\Phi/2}(G_\mu{}^\nu \pm (B_\mu{}^\nu + \frac{1}{T}F_\mu{}^\nu)\partial_\nu \mp iTe^{\Phi/2}(G_{\mu\nu} \pm C_{\mu\nu})X'^\nu \right).$$

Comparison shows that under $\tilde{\mathcal{T}}$ the background fields transform as

$$\begin{aligned} B_{\mu\nu} + \frac{1}{T}F_{\mu\nu} &\to C_{\mu\nu} \\ C_{\mu\nu} &\to B_{\mu\nu} + \frac{1}{T}F_{\mu\nu} \\ G_{\mu\nu} &\to G_{\mu\nu} \\ \Phi &\to -\Phi\,. \end{aligned} \tag{4.33}$$

The fact that under this transformation the NS-NS 2-form is exchanged with what we interpreted as the R-R 2-form and that the dilaton reverses its sign is reminiscent of S-duality. It is well known [26] that T-duality and S-duality are themselves dual under the exchange of the fundamental F-string and the D-string. How exactly this applies to the constructions presented here remains to be investigated. (For instance the sign that distinguishes (4.33) from the expected result would need to be explained, maybe by a change of orientation of the string.)

4.3 Hodge duality on loop space

For the sake of completeness in the following the relation of loop space Hodge duality to the above discussion is briefly indicated. It is found that ordinary Hodge duality is at least superficially related to the algebra isomorphisms discussed in section 4.2 (p. 28). Furthermore a deformed version of Hodge duality is considered which preserves the familiar relation $\mathbf{d}^\dagger = \pm \star\, \mathbf{d} \star^{-1}$.

4.3.1 Ordinary Hodge duality

On a finite dimensional pseudo-riemannian manifold, let $\bar\star$ be the phase-shifted Hodge star operator which is normalized so as to satisfy

$$\begin{aligned} (\bar\star)^\dagger &= -\bar\star \\ (\bar\star)^2 &= 1\,. \end{aligned} \tag{4.34}$$

(For the precise relation of $\bar\star$ to the ordinary Hodge $\star$ see (A.18) of [4].) The crucial property of this operator can be expressed as

$$\bar\star\, \hat{\mathrm{e}}^{\dagger\mu} = \hat{\mathrm{e}}^\mu\, \bar\star\,, \tag{4.35}$$

where $\hat{\mathrm{e}}^{\dagger\mu}$ is the operator of exterior multiplication by dx^μ and $\hat{\mathrm{e}}^\mu$ is its adjoint under the Hodge inner product.

It has been pointed out in [2] that the notion of Hodge duality can be carried over to infinite dimensional manifolds. This means in particular that on loop space there is an idempotent operator $\bar\star$ so that

$$\bar\star\,\mathcal{E}^{\dagger\mu} = \mathcal{E}^{\mu}\,\bar\star \tag{4.36}$$

and

$$\left[\bar\star, X^{(\mu,\sigma)}\right] = 0 = \left[\bar\star, \hat\nabla_{(\mu,\sigma)}\right]\,. \tag{4.37}$$

It follows in particular that the K-deformed exterior derivative is related to its adjoint by

$$\mathbf{d}^{\dagger}{}_{K} = -\bar\star\,\mathbf{d}_K\,\bar\star\,. \tag{4.38}$$

In fact this holds for all the modes:

$$\mathbf{d}^{\dagger}{}^{\dagger}_{K,\xi} = -\bar\star\,\mathbf{d}_{K,\xi}\,\bar\star\,. \tag{4.39}$$

In the spirit of the discussion of T-duality by algebra isomorphisms in section 4.2 (p. 28) one can equivalently say that $\bar\star$ induces an algebra isomorphism $\mathcal{H}$ defined by

$$\mathcal{H}(A) := \bar\star\,A\bar\star\,, \tag{4.40}$$

i.e.

$$\begin{aligned}
\mathcal{H}(-i\partial_\mu) &= -i\partial_\mu\\
\mathcal{H}(X^\mu) &= X^\mu\\
\mathcal{H}\left(\mathcal{E}^{\dagger a}\right) &= \mathcal{E}^{a}\\
\mathcal{H}(\mathcal{E}^{a}) &= \mathcal{E}^{\dagger a}\,.
\end{aligned} \tag{4.41}$$

It is somewhat interesting to consider the result of first applying $\mathcal{H}$ to $\mathbf{d}_K$ and then acting with the deformation operators $\exp(\mathbf{W})$ considered before. This is equivalent to considering the deformation obtained by $\bar\star\,e^{\mathbf{W}}$. This yields

$$\begin{aligned}
\mathbf{d}_K &\to \left(e^{-\mathbf{W}}\bar\star\right)\mathbf{d}_K\left(\bar\star\,e^{\mathbf{W}}\right) = -e^{-\mathbf{W}}\,\mathbf{d}^{\dagger}{}_K\,e^{\mathbf{W}}\\
\mathbf{d}^{\dagger}{}_K &\to \left(e^{\mathbf{W}^\dagger}\bar\star\right)\mathbf{d}^{\dagger}{}_K\left(\bar\star\,e^{-\mathbf{W}^\dagger}\right) = -e^{\mathbf{W}^\dagger}\,\mathbf{d}_K\,e^{-\mathbf{W}^\dagger}\,.
\end{aligned} \tag{4.42}$$

Hence, except for a global and irrelevant sign, the deformations induced by $e^{\mathbf{W}}$ and $\bar\star\,e^{\mathbf{W}}$ are related by

$$\mathbf{W} \leftrightarrow -\mathbf{W}^\dagger\,. \tag{4.43}$$

Looking back at the above results for the backgrounds induced by various $\mathbf{W}$ this corresponds to

$$\begin{aligned}
B &\leftrightarrow C\\
A &\leftrightarrow A\\
\Phi &\leftrightarrow -\Phi\,.
\end{aligned} \tag{4.44}$$

It should be noted though, that unlike the similar correspondence (4.33) both sides of this relation are not unitarily equivalent in the sense that the corresponding superconformal generators $e^{-\mathbf{W}}\bar\star\,\mathbf{d}_K\,\bar\star e^{\mathbf{W}}$ and $e^{-\mathbf{W}}\,\mathbf{d}_K\,e^{\mathbf{W}}$ are not unitarily equivalent.

Nevertheless, it might be that the physics described by both generators is somehow related. This remains to be investigated.

4.3.2 Deformed Hodge duality

The above shows that for general background fields (general deformations of the superconformal algebra) the familiar equality of ${\mathbf{d}^\dagger}^{\mathbf{W}}_{K,\xi}$ with $-\bar{\star}\,\mathbf{d}^{\mathbf{W}}_{K,\xi}\,\bar{\star}^{-1}$ is violated. It is however possible to consider a deformation $\bar{\star}^{\mathbf{W}}$ of $\bar{\star}$ itself which restores this relation:

$$\bar{\star}^{\mathbf{W}} := e^{\mathbf{W}^\dagger}\,\bar{\star}\,e^{\mathbf{W}}\,. \tag{4.45}$$

Obviously this operator satisfies

$${\mathbf{d}^\dagger}^{\mathbf{W}}_{K,\xi} = -\bar{\star}^{\mathbf{W}}\,\mathbf{d}_{K,\xi}\,(\bar{\star}^{\mathbf{W}})^{-1}\,. \tag{4.46}$$

The Hodge star remains invariant under this deformation when $\mathbf{W}$ is anti-Hodge-dual:

$$\bar{\star} = \bar{\star}^{\mathbf{W}} \Leftrightarrow \bar{\star}\,\mathbf{W}\,\bar{\star} = -\mathbf{W}^\dagger\,. \tag{4.47}$$

This is in particular true for the gravitational deformation of section 3.3.1 (p. 16). It follows that $\bar{\star}^{(G)} = \bar{\star}$. This can be understood in terms of the fact that the definition of the Hodge star involves only the orthonormal metric on the tangent space (cf. [4, (A.14)]).

4.4 Deformed inner products on loop space

The above discussion of deformed Hodge duality on loop space motivates the following possibly interesting observation: from the point of view of differential geometry the exterior derivative $\mathbf{d}$ on a manifold is a purely topological object which does not depend in any way on the geometry, i.e. on the metric tensor. The geometric information is instead contained in the Hodge star operator $\star$, the Hodge inner product $\langle\alpha|\beta\rangle = \int\limits_{\mathcal{M}} \alpha\wedge\star\beta$ on differential forms and the adjoint $\mathbf{d}^\dagger$ of $\mathbf{d}$ with respect to $\langle\cdot|\cdot\rangle$.

We have seen in section 4.3.2 (p. 34) that deformations of the Hodge star operator on loop space may encode not only information about the geometry of target space, but also about other background fields, like Kalb-Ramond and dilaton fields. These deformations are accompanied by analogous deformations (3.23) of $\mathbf{d}$ and $\mathbf{d}^\dagger$.

But from this point of view of differential geometry it appears unnatural to associate a deformation of both $\mathbf{d}^\dagger$ as well as $\mathbf{d}$ with a deformed Hodge star operator. One would rather expect that $\mathbf{d}$ remains unaffected by any background fields while the information about these is contained in $\star$, $\langle\cdot|\cdot\rangle$ and $\mathbf{d}^\dagger$.

Here we want to point out that both points of view are equivalent and indeed related by a global similarity transformation ('duality') and that the change in point of view makes an interesting relation to noncommutative geometry transparent.

Namely consider deformed operators

$$\begin{aligned}\mathbf{d}^{(\mathbf{W})} &= e^{-\mathbf{W}}\mathbf{d}e^{\mathbf{W}}\\ \mathbf{d}^{\dagger(\mathbf{W})} &= e^{\mathbf{W}^\dagger}\mathbf{d}^\dagger e^{-\mathbf{W}^\dagger}\end{aligned} \tag{4.48}$$

on an inner product space $\mathcal{H}$ with inner product $\langle\cdot|\cdot\rangle$ as in (3.23).

By applying a global similarity transformation

$$
\begin{aligned}
|\psi\rangle &\to |\tilde{\psi}\rangle := e^{\mathbf{W}}\,|\psi\rangle \\
A &\to \tilde{A} := e^{\mathbf{W}} A e^{-\mathbf{W}}
\end{aligned}
\tag{4.49}
$$

to all elements $|\psi\rangle \in \mathcal{H}$ and all operators A on $\mathcal{H}$ one of course finds

$$
\begin{aligned}
\left(\mathbf{d}^{(\mathbf{W})}\right)^{\tilde{}} &= \mathbf{d} \\
\left(\mathbf{d}^{\dagger(\mathbf{W})}\right)^{\tilde{}} &= e^{\mathbf{W}+\mathbf{W}^\dagger}\mathbf{d}^\dagger e^{-\mathbf{W}-\mathbf{W}^\dagger}\,.
\end{aligned}
\tag{4.50}
$$

By construction, the algebra of $\left(\mathbf{d}^{(\mathbf{W})}\right)^{\tilde{}}$ and $\left(\mathbf{d}^{\dagger(\mathbf{W})}\right)^{\tilde{}}$ is the same as that of $\mathbf{d}^{(\mathbf{W})}$ and $\mathbf{d}^{\dagger(\mathbf{W})}$. But now all the information about the deformation induced by $\mathbf{W}$ is contained in $\left(\mathbf{d}^{\dagger(\mathbf{W})}\right)^{\tilde{}}$ alone. This has the advantage that we can consider a deformed innner product

$$
\langle\cdot|\cdot\rangle_{(\mathbf{W})} := \langle\cdot|\, e^{-(\mathbf{W}+\mathbf{W}^\dagger)}\,\cdot\rangle
\tag{4.51}
$$

on $\mathcal{H}$ with respect to which

$$
\mathbf{d}^{\dagger_{(\mathbf{W})}} = \left(\mathbf{d}^{\dagger(\mathbf{W})}\right)^{\tilde{}}\,,
\tag{4.52}
$$

where $\langle A\cdot|\cdot\rangle_{(\mathbf{W})} := \langle\cdot|A^{\dagger_{(\mathbf{W})}}|\cdot\rangle_{(\mathbf{W})}$. If the original inner product came from a Hodge star this corresponds to a deformation

$$
\star \to \star\, e^{-(\mathbf{W}+\mathbf{W}^\dagger)}\,.
\tag{4.53}
$$

This way now indeed the entire deformation comes from a deformation of the Hodge star and the inner product.

That this is equivalent to the original notion (4.48) of deformation can be checked again by noting that the deformed inner product of the deformed states agrees with the original inner prodcut on the original states

$$
\langle\tilde{\psi}|\tilde{\phi}\rangle_{(\mathbf{W})} = \langle\psi|\phi\rangle\,.
\tag{4.54}
$$

These algebraic manipulations by themselves are rather trivial, but the interesting aspect is that the form (4.51) of the deformation appears in the context of noncommutative spectral geometry [27]. The picture that emerges is roughly that of a spectral triple $(\mathcal{A}, \mathbf{d}_K \pm \mathbf{d}_K^{\dagger(\mathbf{W})}, \mathcal{H})$, where $\mathcal{A}$ is an algebra of functions on loop space (cf. 2.1 (p. 5)), $\mathcal{H}$ is the inner product space of differential forms over loop space equipped with a deformed Hodge inner product (4.51) which encodes all the information of the background fields on target space, and where two Dirac operators are given by $\mathbf{d}_K \pm \mathbf{d}_K^{\dagger(\mathbf{W})}$. There have once been attempts [14, 20] and [28]–[30] to understand the superstring by regarding the worldsheet supercharges as Dirac operators in a spectral triple. Maybe the insight that and how target space background fields manifest themselves as simple algebraic deformations (3.23) of the Dirac operators, or, equivalently, (4.51) of the inner product on $\mathcal{H}$ can help to make progress with this approach.

background field		deformation operator $\mathbf{W}$
metric	$G = E^2$	$\mathcal{E}^\dagger \cdot (\ln E) \cdot \mathcal{E}$
Kalb-Ramond	B	$\frac{1}{2}\mathcal{E}^\dagger \cdot B \cdot \mathcal{E}^\dagger$
dilaton	Φ	$-\frac{1}{2}\Phi\mathcal{E}^\dagger \cdot \mathcal{E}$
2-form for D-string (?)	C	$\frac{1}{2}\mathcal{E} \cdot C \cdot \mathcal{E}$
gauge connection	A	$iA \cdot X'$

Table 1: Deformation operators for various backgrounds.

5. Summary and discussion

We have noted that the loop space formulation of the superstring highlights its Dirac-Kähler structure that again emphasizes the role played by the linear combinations of the leftmoving supercurrent G and its rightmoving counterpart $\tilde{G}$, which are thus seen to be generalized exterior derivative $\mathbf{d}_K$ and coderivative $\mathbf{d}^\dagger{}_K$ on loop space. This fact led us to the study of deformations $\mathbf{d}_K \to e^{-\mathbf{W}}\,\mathbf{d}_K\,e^{\mathbf{W}}$ which preserve the superconformal algebra at the level of Poisson brackets. One important point is that under these deformations the usual supercurrents G and $\tilde{G}$ transform as

$$\left.\begin{matrix} G \\ \tilde{G} \end{matrix}\right\} \propto \mathbf{d}_K \pm \mathbf{d}^\dagger{}_K \to e^{-\mathbf{W}}\,\mathbf{d}_K\,e^{\mathbf{W}} \pm e^{\mathbf{W}^\dagger}\,\mathbf{d}^\dagger{}_K\,e^{-\mathbf{W}^\dagger}\,,$$

so as to preserve the generator of reparameterizations of the string at fixed worldsheet time.

We have shown that the above deformations reproduce, when truncated at first order, the well-known canonical deformations. The hermitean part of the deformation operator $\mathbf{W}$ was found to be the vertex operator of the respective background in the (-1,-1) superghost picture and the anti-hermitean part was seen to give rise to gauge transformations of the background fields.

In the loop-space notation this means that for the NS-NS and NS fields one finds the list of deformation operators in table 1, where $\mathcal{E}^\dagger$ and $\mathcal{E}$ are form creation/annihilation operators on loop space.

Using these deformations the explit (functional/canonical) form of the superconformal generators for all these backgrounds has been obtained, which allowed the study of T-duality by means of algebra isomorphisms. It turned out that under a certain modification of ordinary T-duality the background fields transform as

$$\begin{aligned} C &\leftrightarrow B + \frac{1}{T}F \\ G &\leftrightarrow G \\ \Phi &\leftrightarrow -\Phi\,. \end{aligned}$$

This is reminiscent of S-duality, which is known [26] to be T-duality in the dual string picture. We observe that the 2-form background C might have to be identitfied with the RR 2-form C_2 coupled to the D-string (instead of the F-string).

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The approach presented here allows an interesting perspective on superconformal field theories. In particular it puts the method of canonical deformations of (S)CFTs in a broader context and shows how these have to be generalized beyond leading order. The concise algebraic form in which it expresses the functional form of the constraints of such theories suggests itself for further applications. For instance, as has been discussed in [4], it allows the construction of covariant target space hamiltonians for arbitrary backgrounds, a tool that may be helpful for the study of superstrings in non exactly solvable background fields. In particular, it would be interesting to study how ghosts and spin fields can be incorporated in the present framework and if this would allow to apply the method of [4] to AdS_5 backgrounds. Indications how this should work have been given in [10].

Finally it may help understand the spectral approach to SCFTs, as discussed in section 4.4 (p. 34).

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A. Canonical analysis of bosonic D1 brane action

The bosonic part of the worldsheet action of the D-string is

$$\mathcal{S} = -T\int d^2\sigma\; e^{-\Phi}\sqrt{-\det(G_{ab}+B_{ab}+\frac{1}{T}F_{ab})} + T\int\left(C_2 + C_0(B+\frac{1}{T}F)\right), \tag{A.1}$$

where G, B, C_0 and C_2 are the respective background fields and $F_{ab} = (dA)_{ab}$ is the gauge field on the worldsheet. Indices a, b range over the worldsheet dimensions and indices μ, ν over target space dimensions.

Using *Nambu-Brackets* $\{X^\mu, X^\nu\} := \epsilon^{ab}\partial_a X^\mu\,\partial_b X^\nu$ (with $\epsilon^{01} = 1$, $\epsilon^{ab} = -\epsilon^{ba}$) the term in the square root can be rewritten as

$$-\det\left(G_{ab}+B_{ab}+\frac{1}{T}F_{ab}\right) = -\frac{1}{2}\{X^\mu, X^\nu\}G_{\mu\mu'}G_{\nu\nu'}\{X^{\mu'}, X^{\nu'}\} - (B_{01}+\frac{1}{T}F_{01})^2\,. \tag{A.2}$$

The canonical momenta associated with the embedding coordinates X^μ are

$$\begin{aligned} P_\mu &= \frac{\delta\mathcal{L}}{\delta\dot{X}^\mu} \\ &= T\left(\frac{1}{e^\Phi\sqrt{-\det(G+B+F/T)}}\left(X'^\nu G_{\mu\mu'}G_{\nu\nu'}\{X^{\mu'}, X^{\nu'}\} + B_{\mu\nu}X'^\nu\left(B_{01}+\frac{1}{T}F_{01}\right)\right)\right) + \\ &\quad + T(C_2 + C_0 B)_{\mu\nu}X'^\nu\,. \end{aligned} \tag{A.3}$$

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On the other hand the canonical momenta associated with the gauge field read

$$
\begin{aligned}
E_0 &:= \frac{\delta\mathcal{L}}{\delta\dot{A}_1} = 0 \\
E_1 &:= \frac{\delta\mathcal{L}}{\delta\dot{A}_1} = \frac{1}{e^{\Phi}\sqrt{-\det(G+B+F/T)}}(B_{01} + \frac{1}{T}F_{01}) + C_0 \,.
\end{aligned}
\tag{A.4}
$$

Since the gauge group is U(1), A_μ is a periodic variable and hence the eigenvalues of E_1 are discrete [31]:

$$
E_1 := p \in \mathbb{Z}\,. \tag{A.5}
$$

Inverting (A.4) allows to rewrite the canonical momenta P_μ as

$$
P_\mu = \frac{1}{\sqrt{-\det(G)}}\tilde{T}X'^{\nu}G_{\mu\mu'}G_{\nu\nu'}\{X^{\mu'},X^{\nu'}\} + T(C_2+pB)_{\mu\nu}X'^{\nu}\,, \tag{A.6}
$$

where

$$
\tilde{T} := T\sqrt{e^{-2\Phi} + (p-C_0)^2} \tag{A.7}
$$

is the tension of a bound state of one D-string with p F-strings [32]. In this form it is easy to check that the following two constraints are satisfied:

$$
\begin{aligned}
(P - T(C_2+pB)\cdot X')\cdot(P - T(C_2+pB)\cdot X') + \tilde{T}^2 X'\cdot X' &= 0 \\
(P - T(C_2+pB)\cdot X')\cdot X' &= 0\,,
\end{aligned}
\tag{A.8}
$$

which express temporal and spatial reparameterization invariance, respectively. For constant $\tilde{T}$ this differes from the familiar constraints for the pure F-string only in a redefinition of the tension and the couplings to the background 2-forms.

For non-constant $\tilde{T}$, however, things are a little different. For the purpose of comparison with the results in section 3.3.3 (p. 19) consider the case $B = C_0 = C_2 = p = 0$ and Φ possibly non-constant. In this case the constraints (A.8) can be equivalently rewritten as

$$
{\mathcal{P}_\pm}^2 = 0 \tag{A.9}
$$

with

$$
\mathcal{P}_{\mu,\pm} = e^{\Phi/2}P_\mu \pm Te^{-\Phi/2}G_{\mu\nu}X'^{\nu}\,. \tag{A.10}
$$

Up to fermionic terms this is the form found in (3.47).

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